Problem 1 : Solution/marking scheme – Two Problems in Mechanics *(10 points)*

Part A. The Hidden Disk (3.5 points)

A1 (0.8 pt) Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.

Solution A1: **[0.8]**

Geometric solution: use that torque with respect to point of contact is 0 $\Rightarrow$ center of gravity has to be vertically above point of contact.

$$\sin\phi = \frac{D}{b} \qquad \textbf{0.3}$$

$$\sin\Theta = \frac{D}{r_1} \qquad \textbf{0.3}$$

Here D may be called another name. Solve this:

$$\sin\phi = \frac{r_1}{b}\sin\Theta \ \Rightarrow\ b = \frac{r_1 \sin\Theta}{\sin\phi} \qquad \textbf{0.2}$$

Alternative: Torque and forces with respect to another point: **[0.8]**
Correct equation for torque **0.3**
Correct equation for force **0.3**
Correct solution **0.2**

A2 (0.5 pt) Find the equation of motion for φ. Express the moment of inertia I_S of the cylinder around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that φ is always very small.

Solution A2: **[0.5]**

$$\text{Write some equation of the form } \ddot{\varphi} = -\omega^2\varphi \qquad \textbf{0.1}$$

Writing an equation of the form $\varphi = A\cos\omega t$ is also correct.
Two solutions:

1. Kinetic energy: $\frac{1}{2}I_S\dot{\varphi}^2$ and potential energy: $-bMg\cos\varphi$. Total energy is conserved, and differentiation w.r.t. time gives the equation of motion.
2. Angular equation of motion from torque, $\tau = I_S\ddot{\varphi} = -Mgb\sin\varphi$.

Correct equation (either energy conservation or torque equation of motion) **0.3**
Final answer

$$T = 2\pi\sqrt{\frac{I_S}{Mgb}} \ \Rightarrow\ I_S = \frac{MgbT^2}{4\pi^2} \qquad \textbf{0.1}$$

(Derivation:

$$\Rightarrow \ddot{\varphi} = -\frac{bMg}{I_S}\sin\varphi \simeq -\frac{bgM}{I_S}\varphi$$

so that

$$\omega^2 = \frac{bgM}{I_S}$$

)

A3 (0.4 pt) Find an expression for the distance d as a function of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

Solution A3: **[0.4]**

Some version of the center of mass equation, e.g.

$$b = \frac{dM_2}{M_1 + M_2}$$ **0.2**

correct solution:

$$d = \frac{bM}{\pi h_2 r_2^2(\rho_2 - \rho_1)}$$ **0.2**

A4 (0.7 pt) Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

Solution A4: **[0.7]**

correct answer for moment of inertia of homogeneous disk

$$I_1 = \frac{1}{2}\pi h_1 \rho_1 r_1^4$$ **0.2**

Mass wrong **-0.1**
Factor 1/2 wrong in formula for moment of inertia of a disk **-0.1**
Correct answer for moment of inertia of 'excess' disk:

$$I_2 = \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4$$ **0.2**

Using Steiner's theorem:

$$I_S = I_1 + I_2 + d^2 \pi r_2^2 h_2 (\rho_2 - \rho_1)$$ **0.1**

correct solution:

$$I_S = \frac{1}{2}\pi h_1 \rho_1 r_1^4 + \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4 + \frac{b^2 M^2}{\pi r_2^2 h_2 (\rho_2 - \rho_1)}$$ **0.2**

In terms of d rather than b gives 0.1pts rather than 0.2pts for the final answer:

$$I_S = \frac{1}{2}\pi h_1 \rho_1 r_1^4 + \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4 + d^2 \pi r_2^2 h_2 (\rho_2 - \rho_1)$$ **0.1**

A5 (1.1 pt) Using all the above results, write down an expression for h_2 and r_2 in terms of b, T and the quantities (1). You may express h_2 as a function of r_2.

Solution A5: **[1.1]**

It is not clear how exactly students will attempt to solve this system of equations. It is likely that they will use the following equation:

$$M = \pi r_1^2 h_1 \rho_1 + \pi r_2^2 h_2 (\rho_2 - \rho_1) \ .$$ **0.3**

solve I_S for r_2^2:

$$r_2^2 = \frac{2}{M - \pi r_1^2 h_1 \rho_1}\left(I_S - \frac{1}{2}\pi h_1 \rho_1 r_1^4 - b^2 \frac{M^2}{M - \pi r_1^2 h_1 \rho_1}\right)$$ **0.4**

replace I_S by T:

$$I_S = \frac{MgbT^2}{4\pi^2}$$ **0.1**

solve correctly for r_2:

$$r_2 = \sqrt{\frac{2}{M - \pi r_1^2 h_1 \rho_1}\left(M\frac{bgT^2}{4\pi^2} - \frac{1}{2}\pi h_1 \rho_1 r_1^4 - b^2 \frac{M^2}{M - \pi r_1^2 h_1 \rho_1}\right)}$$ **0.1**

write down an equation for h_2 along the lines of $M = \pi r_1^2 \rho_1 h_1 + \pi r_2^2 (\rho_2 - \rho_1) h_2$ and solve it correctly:

$$h_2 = \frac{M - \pi r_1^2 \rho_1 h_1}{\pi r_2^2 (\rho_2 - \rho_1)}$$ **0.2**

Part B. Rotating Space Station (6.5 points)

B1 (0.5 pt) At what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth's surface?

Solution B1: **[0.5]**

An equation for the centrifugal force along the lines of

$$F_{ce} = m\omega^2 r$$ **0.1**

Balancing the forces, correct equation

$$g_E = \omega_{ss}^2 R$$ **0.2**

Correct solution

$$\omega_{ss} = \sqrt{g_E/R}$$ **0.2**

B2 (0.2 pt) Assuming that on Earth gravity is constant with acceleration g_E, what would be the angular oscillation frequency ω_E that a person on Earth would measure?

Solution B2: **[0.2]**

Realize that result is independent of g_E **0.1**

Correct result:

$$\omega_E = \sqrt{k/m}$$ **0.1**

B3 (0.6 pt) What angular oscillation frequency ω does Alice measure on the space station?

Solution B3: **[0.6]**

some version of the correct equation for force

$$F = -kx \pm m\omega_{ss}^2 x$$ **0.2**

getting the sign right

$$F = -kx + m\omega_{ss}^2 x$$ **0.2**

Find correct differential equation

$$m\ddot{x} + (k - m\omega_{ss}^2)x = 0$$ **0.1**

Derive correct result

$$\omega = \sqrt{k/m - \omega_{ss}^2}$$ **0.1**

Using g_E/R instead of ω_{ss}^2 is also correct.

B4 (0.8 pt) Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ (linear approximation is enough). The radius of the Earth is given by R_E.

Solution B4: **[0.8]**

$$g_E(h) = -GM/(R_E + h)^2 \qquad \mathbf{0.1}$$

linear approximation of gravity:

$$g_E(h) = -\frac{GM}{R_E^2} + 2h\frac{GM}{R_E^3} + \dots \qquad \mathbf{0.2}$$

Realize that $g_E = GM/R_E^2$:

$$g_E(h) = -g_E + 2hg_E/R_E + \dots \qquad \mathbf{0.1}$$

Opposite sign is also correct, as long as it is opposite in both terms.
Realize what this means for force, i.e. that the constant term can be eliminated by shifting the equilibrium point:

$$F = -kx + 2xmg_E/R_E \qquad \mathbf{0.2}$$

Find correct differential equation

$$m\ddot{x} + (k - 2mg_E/R_E)x = 0 \qquad \mathbf{0.1}$$

correct result

$$\tilde{\omega}_E = \sqrt{k/m - 2g_E/R_E} \qquad \mathbf{0.1}$$

No points are deducted if student answers with $\tilde{\omega}_E/(2\pi)$ because ”oscillation frequency” might also be interpreted as inverse period.

B5 (0.3 pt) For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the surface of the Earth? Express your answer in terms of R_E.

Solution B5: **[0.3]**

Write down equation

$$\omega_{ss}^2 = 2g_E/R_E \qquad \mathbf{0.1}$$

Solve

$$R = R_E/2 \qquad \mathbf{0.2}$$

If GM/R_E^2 rather than g_E is used, give only *0.1pt*.

B6 (1.1 pt) Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.

Solution B6: [1.1]

There are several possible solutions.

Solution one – Using Coriolis force

- Velocity v_x

  Equation for Coriolis force with correct velocity:

  $$F_C(t) = 2m\omega_{ss}^2 Rt\omega_{ss} = 2m\omega_{ss}^3 Rt$$ **0.1**

  Integrate this, or realize that it is like uniform acceleration for the velocity:

  $$v_x(t) = \omega_{ss}^3 Rt^2$$ **0.2**

  plug in correct value for

  $$t = \sqrt{2H/\omega_{ss}^2 R}$$ **0.2**

  overall correct result

  $$v_x = 2H\omega_{ss}$$ **0.1**

- The displacement d_x:

  Integrate $v_x(t)$:

  $$d_x = \frac{1}{3} R\omega_{ss}^3 t^3$$ **0.3**

  Instead of integrating, students may simply 'average' by taking $\frac{1}{2}$ of the final velocity. This gives a factor of $\frac{1}{2}$ instead of $\frac{1}{3}$. *Deduct a total of 0.1 pts for this.* **-0.1**

  Plug in value for t

  $$d_x = \frac{1}{3} R\omega_{ss}^3 (2H/\omega_{ss}^2 R)^{3/2} = \frac{1}{3} 2^{3/2} H^{3/2} R^{-1/2} = \frac{1}{3}\sqrt{\frac{8H^3}{R}}$$ **0.2**

Solution two – Using inertial frame This solution is similar to the way to solve B7, but needs more complicated approximations than Solution one.

- v_x

  Here ϕ denotes the angle swept by the mass and α the angle the astronauts (and tower) has rotated when the mass lands on the floor, see

  Initially the velocity of the mass in an inertial frame is $v_x = \omega_{ss}(R - H)$. **0.1**

  When the mass lands, the x-direction has been rotated by ϕ so the new horizontal velocity component is then

  $$\omega_{ss}(R - H)\cos\phi$$ **0.1**

  (Student may also write $\cos\alpha$ instead of $\cos\phi$, since $d_x \ll H$.)

  $$\cos\phi = \frac{R - H}{R} = 1 - \frac{H}{R}$$ **0.1**

  Transforming to the rotating reference frame, one needs to subtract $\omega_{ss}R$. **0.1**

  Finally in the reference frame of the astronauts

  $$v_x = \omega_{ss}R\left(1 - \frac{H}{R}\right)^2 - \omega_{ss}R \approx \omega_{ss}R\left(1 - 2\frac{H}{R}\right) - \omega_{ss}R = -2\omega_{ss}H$$ **0.2**

  The sign of the velocity depend on the choice of reference direction, so a positive sign is also correct.

- d_x

  With the notation from the calculation of v_x

$$d_x = (\alpha - \phi)R \qquad \textbf{0.1}$$

$$\phi = \arccos\left(1 - \frac{H}{R}\right)$$

$$\alpha = \omega_{ss} t$$

  where t is the fall time of the mass, which is given by

$$t = \frac{\sqrt{R^2 - (R-H)^2}}{\omega_{ss}(R-H)} \qquad \textbf{0.1}$$

  (see solution to B7)

  Writing $\xi \equiv H/R$ this means

$$d_x = \left[\frac{\sqrt{1-(1-\xi)^2}}{1-\xi} - \arccos(1-\xi)\right] R \qquad \textbf{0.3}$$

  which is a valid end answer to the problem. It is possible, but not necessary, to approximate this for small ξ:

$$\arccos(1-\xi) \approx \sqrt{2\xi}\left(1 + \frac{\xi}{12}\right)$$

  which after insertion into the equation for d_x and approximation of small ξ yields the same result as in Solution one:

$$d_x = \frac{2}{3}\sqrt{\frac{2H^3}{R}}$$

  If this end answer misses the factor 2/3, deduct 0.1 points. **-0.1**

Solution three – Inertial frame with geometry trick

This is an alternative solution to obtain d_x

The mass travels the distance l, and during the fall the space station rotates by ϕ, see Figure 2. According to the intersecting chord theorem,

$$l^2 = H(2R - H) \qquad \textbf{0.1}$$

The rotated angle is $\phi = \omega_{ss} t$ where

$$t = \frac{l}{R - H} \qquad \textbf{0.1}$$

is the fall time. Thus

$$\phi = \frac{\sqrt{H(2R-H)}}{R-H} \qquad \textbf{0.1}$$

$$\frac{d}{R} = \phi - \arcsin\frac{l}{R} = \frac{\sqrt{H(2R-H)}}{R-H} - \arcsin\sqrt{x(2-x)} \qquad \textbf{0.1}$$

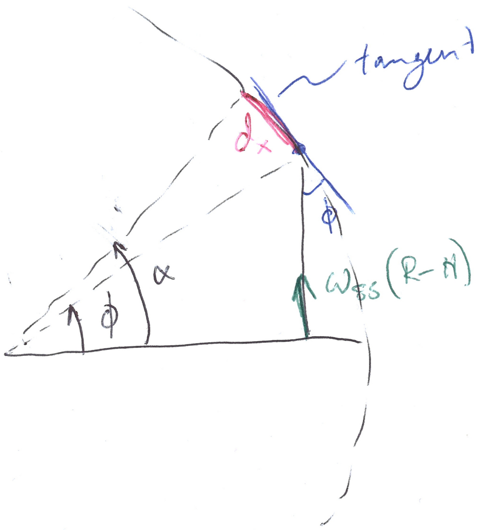


Figure 1: Notation for solution two

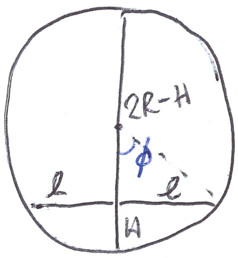


Figure 2: Notation for solution three.

Denote $x \equiv H/R$ and $y \equiv \sqrt{x(2-x)}$. Since

$$\arcsin y \approx y + \frac{y^3}{6}$$

one gets

$$\frac{d}{R} \approx y(1+x) - y - y^3/6 = y(x - y^2/6) \approx 2xy/3 \approx 2x\sqrt{2x}/3 = \frac{2}{3}\sqrt{\frac{2H^3}{R}}$$

Final answer **0.1**

B7 (1.3 pt) Find a lower bound for the height of the tower for which it can happen that $d_x = 0$.

Solution B7: **[1.3]**

The key is to use a non-rotating frame of reference. If the mass is released close enough

to the center, its linear velocity will be small enough for the space station to rotate more than 2π before it hits the ground.
The velocity is given by

$$v = \omega_{ss}(R - H)$$ **0.1**

distance d that the mass flies before hitting the space station

$$d^2 = R^2 - (R - H)^2$$ **0.1**

use non-rotating frame of reference to obtain time t until impact

$$t = d/v = \frac{\sqrt{R^2 - (R-H)^2}}{\omega_{ss}(R - H)}$$ **0.1**

Now there are several possible ways to relate H and the rotated angle ϕ of the space station:

Solution one

$$t = \frac{R \sin\phi}{\omega_{ss} R \cos\phi}$$ **0.2**

This time must match $t = \phi/\omega_{ss}$. Obtain the equation

$$\phi = \tan\phi$$ **0.2**

Realizing that there is an infinite number of solutions. **0.2**

This equation has one trivial solution $\phi = 0$, next solution is slightly less than $3\pi/2$ which corresponds to the case $H > R$ (and is thus not correct). The one that gives a lower bound for H is the third solution

$$\phi \approx 5\pi/2$$

The equation $\phi = \tan\phi$ can be solved graphically or numerically to obtain a close value ($\phi = 7.725$ rad) which means

$$H/R = (1 - \cos\phi) \approx 0.871$$

Give points if the method is correct, depending on the value of H/R found, according to these intervals: **0.4**

$0.85 \leq H/R \leq 0.88$: *0.4 pts*
$0.5 \leq H/R < 0.85$: *0.3 pts*
$0 < H/R < 0.5$ or $H > 0.88$: *0.2 pts*
$H = 0$ or method is incorrect: *0 pts*

Solution two

relation between H and rotated angle ϕ

$$\frac{R - H}{R} = \cos\phi$$ **0.2**

obtain equation of the form

$$\frac{H}{R} = 1 - \cos\left(\frac{\sqrt{1 - (1 - H/R)^2}}{1 - H/R}\right)$$ **0.2**

Figure 3 gives a plot of $f(x) = 1 - \cos\left(\frac{\sqrt{1-(1-x)^2}}{1-x}\right)$. The goal is to find an approximate solution for the second intersection. The first intersection is discarded – it is introduced because of $\cos\phi = \cos(-\phi)$ and corresponds to a situation with $H > R$.
Realizing that there is an infinite number of solutions. **0.2**

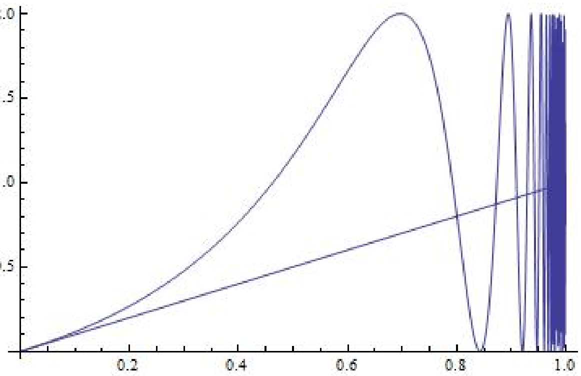


Figure 3: Plot of $f(H/R)$ and H/R

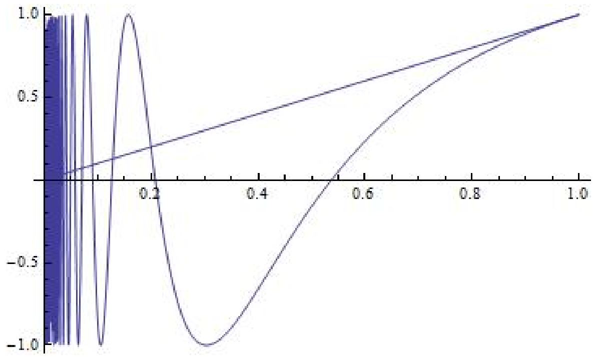


Figure 4: Plot of $g(x)$ and x

- introduce new variable $x := 1 - H/R$, so that the equation becomes

$$x = \cos(\sqrt{1-x^2}/x) =: g(x)$$

- $g(x)$ is then smaller than x up to the first solution. In particular it is negative in some region (see figure 4). Finding the third zero thus gives a lower bound for the solution:

$$\frac{\sqrt{1-x^2}}{x} = 5\pi/2$$

- give lower bound

$$x = 1/\sqrt{25\pi^2/4+1} \Rightarrow H = R(1 - 1/\sqrt{25\pi^2/4+1}) \approx 0.874$$

Note: the actual result is $H/R = 0.871\ldots$.

Use the same points for the numerical answer as was mentioned in solution one. **0.4**

If the student plots f rather than g, find solution to $f = 1$: is equivalent to the solution above. *Give same number of points.*

It is also possible to use $\cos\left(\frac{\sqrt{1-x^2}}{x}\right) = \sin(1/x)$.

B8 (1.7 pt) Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 4).

- Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss}d$ is small.
- Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.

Solution B8: **[1.7]**

Note: we did not specify the overall sign of the Coriolis force. Give same amount of points if using opposite convention, but it has to be consistent! Otherwise: subtract *0.1pt* for each instance of inconsistency. **-0.1**

Students are allowed to express everything in terms of ω, they don't need to write $\sqrt{k/m - \omega_{ss}^2}$ explicitly. Deduct *0.1pt* however if they use k/m instead of ω.. **-0.1**

Realize that $y(t)$ is standard harmonic oscillation:

$$y(t) = A\cos\omega t + B$$ **0.1**

Give correct constants from initial conditions

$$y(t) = -d\cos\omega t$$ **0.2**

Correct expression for $v_y(t)$:

$$v_y(t) = -d\omega\sin\omega t$$ **0.1**

Coriolis force in x-direction

$$F_x(t) = 2m\omega_{ss}v_y(t) = -2m\omega_{ss}d\omega\sin\omega t$$ **0.2**

Realize that this implies that $x(t)$ is also a harmonic oscillation... **0.1**

...but with a constant movement term superimposed: vt **0.1**

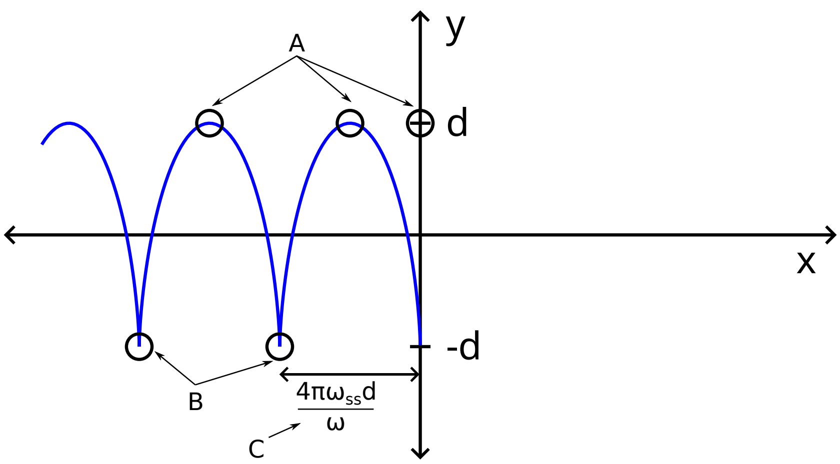
getting the correct amplitude:

$$A = \frac{2\omega_{ss}d}{\omega}$$ **0.1**

Correct answer with correct initial conditions:

$$x(t) = \frac{2\omega_{ss}d}{\omega}\sin\omega t - 2\omega_{ss}dt$$ **0.2**

Sketch:



Correct qualitative sketch:	
periodic motion	**0.1**
overall constant movement	**0.1**
B): cusps	**0.1**
And additionally correct quantitative sketch:	
A)+B): peaks and cusps are at $y = \pm d$	**0.1**
C): cusps are at distance $\Delta x = \dfrac{4\pi\omega_{ss}d}{\omega}$ from each other	**0.2**