Problem 2: Viscoelasticity of a polymer thread (10 points)

Part A. Stress-relaxation measurements (1.9 points)

A.1

Measurement: $\ell_0 = 42.7 + 2 \times 0.5 = 43.7$ cm ,

A.1 0.3pt

$\ell_0 = (43.7 \pm 0.2)$ cm .

A.2

A.2 0.3pt

$P_0 = (81.11 \pm 0.03)$ gf .

A.3

The table contains the readings on the scale P (Question A.3) and the force on the thread, $F(t)$, at constant strain (Question D.1). The values of $\frac{dF}{dt}$ (Question D.6) were computed numerically using equal time intervals. The function $y(t)$ is given by $y(t) = F(t) - F_0 - F_1\,e^{-t/\tau_1}$ (Question D.10).

A.3 1.0pt

t /s	$P(t)$ /gf	F /gf	$\frac{dF}{dt}$ /gf s^{-1}	$y(t)$ /gf
10	35.7	45.41		2.82
17	36.2	44.91		2.33
26	36.6	44.51		1.95
32	36.8	44.31		1.76
40	37.0	44.11		1.57
46	37.1	44.01		1.48
51	37.2	43.91		1.38
58	37.3	43.81		1.29
65	37.4	43.71		1.20

A.3 0.3pt

t /s	$P(t)$ /gf	F /gf	$\frac{dF}{dt}$ /gf s^{-1}	$y(t)$ /gf
73	37.5	43.61		1.12
84	37.6	43.51		1.03
94	37.7	43.41		0.94
105	37.8	43.31		0.86
118	37.9	43.21		0.77
136	38.0	43.11		0.70
151	38.1	43.01		0.62
173	38.2	42.91		0.55
193	38.3	42.81		0.48
217	38.4	42.71		0.41
247	38.5	42.61		0.35
279	38.6	42.51		0.29
317	38.7	42.41		0.23
358	38.8	42.31		0.18
408	38.9	42.21		0.14
471	39.0	42.11		0.11
525	39.1	42.01		0.07
591	39.2	41.91		0.03
600	39.2	41.91		0.04
672	39.3	41.81		0.01
773	39.4	41.71		0.007
866	39.5	41.61		−0.01
900	39.52	41.59		−0.00
993	39.6	41.51		−0.01
1124	39.7	41.41		
1200	39.74	41.37	-7.00×10^{-4}	
1272	39.8	41.31		
1419	39.9	41.21		
1500	39.94	41.17	-5.33×10^{-4}	
1628	40.0	41.11		
1800	40.06	41.05	-4.67×10^{-4}	
1869	40.1	41.01		
2037	40.2	40.91		
2100	40.22	40.89	-3.83×10^{-4}	
2400	40.29	40.82		

A.4

Measurement: $\ell = 50.0 + 2 \times 0.5 = 51.0$ cm ,

A.4 0.3pt

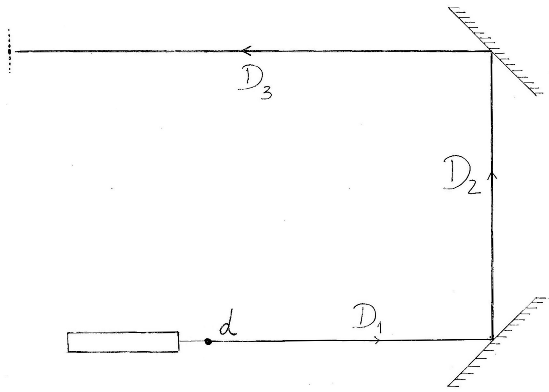
$$\ell = (51.0 \pm 0.2) \text{ cm} .$$

Part B. Measurement of the streched thread diameter (1.5 points)

B.1

Two mirrors are used to maximize the distance D and consequently the distance between diffraction minima.

B.1 Sketch of the method 0.6pt



B.2

The total distance D is the sum

$$D = D_1 + D_2 + D_3 = (26.0 + 36.0 + 102.3) \text{ cm} = 164.3 \text{ cm} = 1.643 \text{ m} .$$

The estimated uncertainties are

$$\sigma_{D_1} = \sigma_{D_2} = \sigma_{D_3} \approx 0.5 \text{ cm} \Rightarrow \sigma_D = \sqrt{3 \times \sigma_{D_1}^2} = 0.5 \times \sqrt{3} = 0.87 \text{ cm} .$$

B.2 0.3pt

$$D = (1.643 \pm 0.009) \text{ m} .$$

B.3

The distance between minima, x, is quite small. To reduce the error, the total distance Nx, with $N = 22$, was measured:

$$22\,x = 49\,\text{mm} \;\Rightarrow\; \bar{x} = 2.227\,\text{mm}\,.$$

The corresponding uncertainty is

$$\sigma_{\bar{x}} = \frac{\sigma_{22\,x}}{22} = \frac{0.25\,\text{mm}}{22} = 0.011\,\text{mm}\,.$$

B.3	$\bar{x} = (2.227 \pm 0.011)\,\text{mm}\,.$	0.3pt

B.4

Using previous results, we get

$$d = \frac{\lambda}{\sin\theta} \simeq \frac{\lambda D}{\bar{x}} = \frac{650\times10^{-9}\,\text{m}\times1.643\,\text{m}}{2.227\times10^{-3}\,\text{m}} = 4.795\times10^{-4}\,\text{m} = 0.480\,\text{mm}\,.$$

For the uncertainties, we have

$$\frac{\sigma_d}{d} = \frac{\sigma_\lambda}{\lambda} + \frac{\sigma_D}{D} + \frac{\sigma_{\bar{x}}}{\bar{x}} = \frac{10}{650} + \frac{0.0087}{1.643} + \frac{0.011}{2.227} = 0.02517 \;\Rightarrow\; \sigma_d = 0.02517\times0.480\,\text{mm} = 0.012\,\text{mm}\,.$$

B.4	$d = (0.480 \pm 0.012)\,\text{mm}\,.$	0.3pt

Part C. Change to a new thread (0.3 points)

C.1

Measurement: $\ell_0' = 31.6 + 2\times0.5 = 32.6\,\text{cm}$.

C.1	$\ell_0' = (32.6 \pm 0.2)\,\text{cm}\,.$	0.3pt

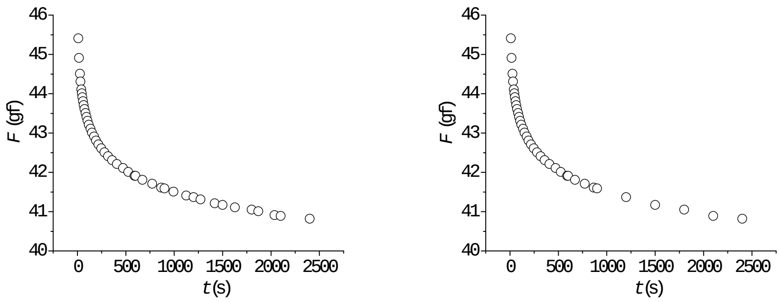
Part D. Data Analysis (5.7 points)

D.1

The force on the thread was calculated as $F(t) = (P_0 - P(t))$, in gram-force units.

D.1	See column $F(t)$ in the table in A.3.	0.3pt

D.2



D.2 0.4pt

Left: $F(t)$ sampled at unequal time intervals. **Right**: $F(t)$ sampled at equal time intervals for $t > 1000$ s.

D.3

The dimensionless quantity ϵ is given by

$$\epsilon = \frac{\ell - \ell_0}{\ell_0} = \frac{51.0 - 43.7}{43.7} = 0.167 \ .$$

The uncertainty in ϵ, σ_ϵ, is calculated propagating the uncertainties in the measured length, σ_ℓ and σ_{ℓ_0}:

$$\begin{aligned} \frac{\sigma_\epsilon}{\epsilon} &= \frac{\sigma_{(\ell-\ell_0)}}{\ell - \ell_0} + \frac{\sigma_{\ell_0}}{\ell_0} \\ &= \frac{\sqrt{\sigma_\ell^2 + \sigma_{\ell_0}^2}}{\ell - \ell_0} + \frac{\sigma_{\ell_0}}{\ell_0} \\ &= \frac{0.2 \times \sqrt{2}}{7.3} + \frac{0.2}{43.7} \\ &= 0.0433 \end{aligned}$$

Therefore, $\sigma_\epsilon = 0.0433 \times 0.167 = 0.0072$.

D.3	$\epsilon = 0.167 \pm 0.007$.	0.3pt

D.4

One has

$$\frac{\sigma}{\epsilon} = \frac{F}{\epsilon\, S} \, .$$

In this case, $S = \pi(d/2)^2 = 1.809 \times 10^{-7}\,\text{m}^2$ and $\epsilon = 0.167$. We also have $1\,\text{gf} = g \times 10^{-3}\,\text{N}$ with $g = 9.8\,\text{m}\,\text{s}^{-2}$. Therefore, if F is in gram-force units we have

$$\frac{\sigma}{\epsilon} = \frac{9.8 \times 10^{-3}\,\text{gf}^{-1}\,\text{N}}{0.167 \times 1.809 \times 10^{-7}\,\text{m}^2}\; F = \left(324293\,\text{gf}^{-1}\,\text{N}\,\text{m}^{-2}\right)\; F \, ,$$

where F is in gf , and σ is in $\text{N}\,\text{m}^{-2}$. Comparing with $\frac{\sigma}{\epsilon} = \beta\, F$ we get

$$\beta = 324293\,\text{gf}^{-1}\,\text{N}\,\text{m}^{-2} \quad .$$

Note that, if we write

$$F(t) = F_0 + F_1\,\text{e}^{-t/\tau_1} + F_2\,\text{e}^{-t/\tau_2} + F_3\,\text{e}^{-t/\tau_3} + \cdots \tag{1}$$

and compare with equation

$$\frac{\sigma}{\epsilon} = \beta\, F(t) = E_0 + E_1\,\text{e}^{-t/\tau_1} + E_2\,\text{e}^{-t/\tau_2} + E_3\,\text{e}^{-t/\tau_3} + \cdots \tag{2}$$

we conclude that $E_0 = \beta F_0$, $E_1 = \beta F_1$, $E_2 = \beta F_2$, etc.

D.4 0.3pt

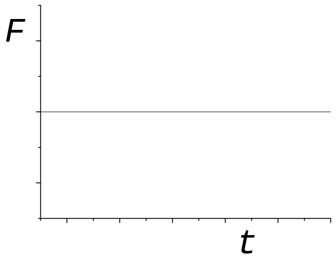
$$\beta = 3.24 \times 10^5\,\text{gf}^{-1}\,\text{N}\,\text{m}^{-2} \quad .$$

D.5

For a purely elastic process, $\sigma = \epsilon\, E_0$ and

$$F = \alpha\, \sigma = \alpha\, \epsilon\, E_0 \quad .$$

Thus, a graph of a constant function is expected.



D.5 0.4pt

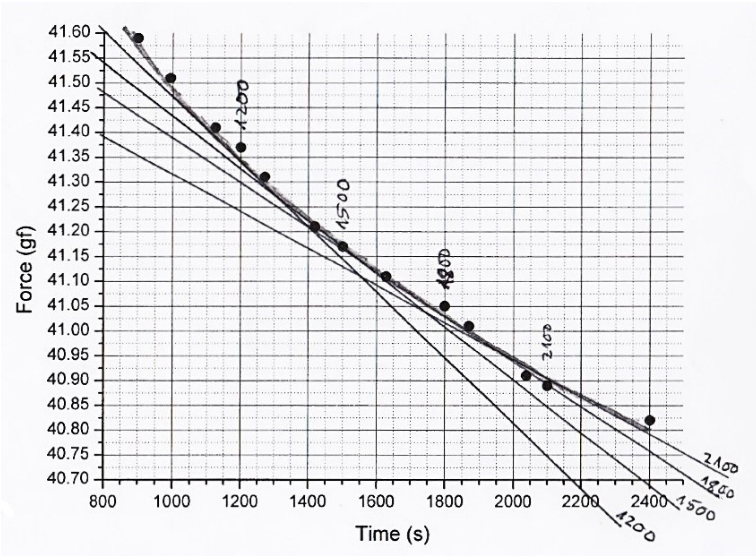
D.6

The data for $\frac{dF}{dt}$ inserted in table introduced in A.3, was computed numerically for equal time intervals.

However, the graphical method is also exemplified. In the present graph, tangent lines to $F(t)$ are drawn at four different time instants (1200,1500,1800 and 2100 s). The slopes of those lines are a measure of $\frac{dF}{dt}$ at those instants.

D.6 See in the table used in A.3, the column with $\frac{dF}{dt}$. 0.5pt

This graph is present only if a graphical method is used.



D.7

For a single viscoelastic process,

$$F = \frac{1}{\beta}\left(E_0 + E_1\,\mathrm{e}^{-t/\tau_1}\right) = F_0 + F_1\,\mathrm{e}^{-t/\tau_1}\ .$$

Therefore,

D.7 0.3pt

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -\frac{F_1}{\tau_1}\,\mathrm{e}^{-t/\tau_1}\ , \qquad \text{where} \qquad F_1 = \frac{E_1}{\beta}\ .$$

D.8

The linearisation of the expression of dF/dt is accomplished using logarithms:

$$\ln\left(-\frac{\mathrm{d}F}{\mathrm{d}t}\right) = \ln\left(\frac{F_1}{\tau_1}\right) - \frac{1}{\tau_1}\,t\ .$$

The plot of ln$(-\mathrm{d}F/\mathrm{d}t)$ is shown in the graph below for a case where the derivative was obtained numerically (left) and using a graphic method (right).

For the left graph, the best straight line is ln$(-\mathrm{d}F/\mathrm{d}t) = m_1\, t + b_1$ where $m_1 = (-6.47 \pm 0.62) \times 10^{-4}$ and $b_1 = (-6.52 \pm 0.11)$, using t in seconds and the force in gram-force units. If the derivative is computed numerically for unequal time intervals, the final parameters E_1 and τ_1 are similar.

The best straight line for the right graph yields $m_1 = (-6.00 \pm 0.15) \times 10^{-4}$ and $b_1 = (-6.63 \pm 0.02)$ using t in seconds and the force in gram-force units.

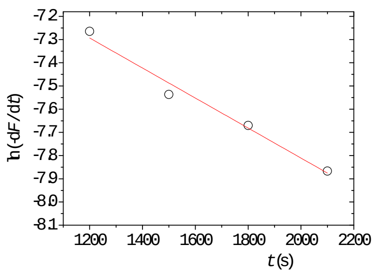
Thus, using the data from the left graph, $\tau_1 = \frac{1}{-m_1} = 1546\,\mathrm{s}$ and

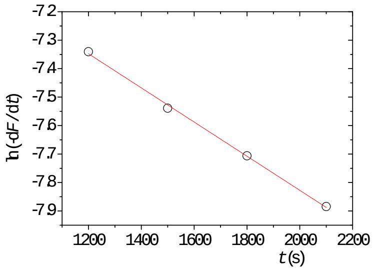
$$F_1 = \tau_1\, \mathrm{e}^{b_1} = 2.28\,\mathrm{gf} \Rightarrow E_1 = \beta\, F_1 = 7.39 \times 10^5\,\mathrm{N\,m^{-2}} \quad .$$

For the right graph, the final parameters are $\tau_1 = 1667\,\mathrm{s}$ and $E_1 = 7.13 \times 10^5\,\mathrm{N\,m^{-2}}$.

D.8 1.0pt

$$\tau_1 = 1546\,\mathrm{s}\,, \qquad E_1 = 7.39 \times 10^5\,\mathrm{N\,m^{-2}}\,.$$





Left: $\frac{\mathrm{d}F}{\mathrm{d}t}$ computed numerically using equal time intervals. **Right:** using data from the graph in D.6.

D.9

For the 4 points on the left graph in D.8, we can write

$$F(t) = F_0 + F_1\, \mathrm{e}^{-t/\tau_1} \;\Rightarrow\; F_0 = F(t) - F_1\, \mathrm{e}^{-t/\tau_1}$$

Thus, averaging F_0 for the 4 points of the left graph in D.8:

$$F_0 = \left(\frac{40.32 + 40.31 + 40.34 + 40.30}{4}\right) = 40.32\,\mathrm{gf}$$

Finally,

$$E_0 = \beta\, F_0 = 324293 \times 40.32\;\; \mathrm{N\,m^{-2}}\,.$$

D.9 0.3pt

$$E_0 = 1.31 \times 10^7\,\mathrm{N\,m^{-2}}\,.$$

D.10

The function $y(t)$ is given by

$$y(t) = F(t) - F_0 - F_1\,\mathrm{e}^{-t/\tau_1}\ ,$$

and was added in the Table introduced in A.3 using $F_0 = 40.32$ gf , $F_1 = 2.28$ gf and $\tau_1 = 1546$ s.

D.10	See column $y(t)$ in the Table in A.3.	0.3pt

D.11

Since

$$y(t) = F(t) - F_0 - F_1\,\mathrm{e}^{-t/\tau_1}\ ,$$

then

$$y(t) = F_2\,\mathrm{e}^{-t/\tau_2} + F_3\,\mathrm{e}^{-t/\tau_3} + \cdots \quad , \quad \tau_2 > \tau_3 > \cdots$$

At long times, when the contributions from the higher components are small enough, we expect a linear behaviour for $\ln y(t)$:

$$\ln y = \ln F_2 - \frac{1}{\tau_2}\,t\ .$$

In this case, the $y(t)$ data points become meaningless above 500 s. In the region 200-500 s the graph is linear and that region can be used to extract the parameters of the second component. The equation of the straight line is $\ln y_2 = b_2 + m_2\,t$. From the graph below,

$$m_2 = -(5.65 \pm 0.19) \times 10^{-3} \ \Rightarrow\ \tau_2 = \frac{1}{-m_2} = 177\,\mathrm{s}$$

$$b_2 = 0.33 \pm 0.07 \ \Rightarrow\ F_2 = \mathrm{e}^{b_2} = 1.39 \ \Rightarrow\ E_2 = \beta\,F_2 = 4.5 \times 10^5\ \mathrm{N\,m^{-2}}\ .$$

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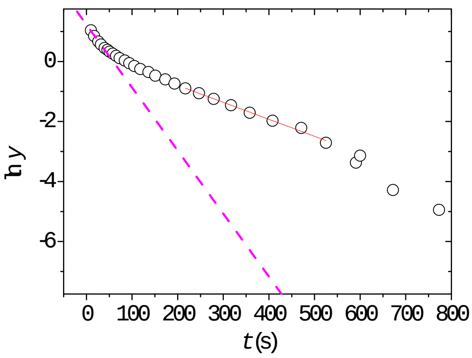
D.11 1.0pt

$$E_2 = 4.5 \times 10^5\ \mathrm{N\,m^{-2}}\ , \qquad \tau_2 = 177\,\mathrm{s}\ .$$



The best straight line in the range 200-500 s yield the parameters τ_2 and E_2 (Question D.11). The slope of the best straight line in the range $[10, 30]$ s give an estimate of τ_3 (Questions D.12 and D.13).

D.12

Below around 30 s there is clear deviation from a linear behaviour indicating the presence of a third component. In our case, the first data point was acquired at $t = 10$ s.

D.12 (0.3 pt)

$$t_i = 10\,\text{s} \quad , \quad t_f = 30\,\text{s}$$

D.13

Drawing a line in the graph using the first data points (in the range defined in D.12), as shown in the graph in D.11, τ_3 can be estimated as:

$$m_3 = -0.02 \;\Rightarrow\; \tau_3 \approx m_3^{-1} \;,$$

D.13 0.3pt

$$\tau_3 \approx 50\,\text{s} \;.$$

Part E. Measuring E in constant stress conditions (0.6 points)

E.1

From Question C.1 we have

$$\ell'_0 = (32.60 \pm 0.2)\,\text{cm} \;.$$

The final length ℓ' should be measured. In our case,

$$\ell' = 42.2 + 2 \times 0.5 = 43.2\,\text{cm} \;\Rightarrow\; \ell' = (43.2 \pm 0.2)\,\text{cm} \;.$$

Therefore, the strain is

$$\epsilon = \frac{\ell' - \ell'_0}{\ell'_0} = 0.325 \;.$$

Given that

$$E = \frac{\sigma}{\epsilon} = \frac{\frac{Mg}{\pi R^2}}{\epsilon} = \frac{80.2 \times 10^{-3} \times 9.8}{\pi \times (0.24 \times 10^{-3})^2 \times 0.325} = 1.337 \times 10^7\,\text{N}\,\text{m}^{-2} \;.$$

Note that the radius R of the stretched thread was not measured. We used the value measured in task B.4: $R \approx 0.24 \times 10^{-3}$ m.

The relative difference is

$$\frac{E - E_0}{E_0} = 0.021 \; .$$

E.1	$$E = 1.337 \times 10^7\,\mathrm{N\,m^{-2}} \quad , \quad \frac{E - E_0}{E_0} = 2.1\% \; .$$	0.6pt