

## Physics of Live Systems (10 points)

### Part A. The physics of blood flow (4.5 points)

#### A.1

Since the vessel network is symmetrical, the flow in a vessel of level  $i + 1$  is half the flow in a vessel of level  $i$ .

In this way, we can sum the pressure differences in all levels:

$$\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.$$

Introducing the radii dependences yields

$$\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^i 2^{i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4}.$$

Therefore

$$Q_0 = \Delta P \frac{\pi r_0^4}{8N\ell_0 \eta}.$$

Hence, the flow rate for a vessel network in level  $i$  is

**A.1**

1.3pt

$$Q_i = \Delta P \frac{\pi r_0^4}{2^{i+3} N \ell_0 \eta}.$$

#### A.2

Replace values in the formula and change units appropriately

$$\begin{aligned} Q_0 &= \frac{\Delta P \pi r_0^4}{8N\ell_0 \eta} = \\ &= \frac{(55 - 30) \times 1.013 \times 10^5 \times 3.1415 \times (6.0 \times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0 \times 10^{-10} \text{ m}^3/\text{s} \end{aligned}$$

to obtain the final value in the requested unites:

**A.2**

0.5pt

$$Q_0 \simeq 1.5 \text{ ml/hr}.$$

## A.3

The current is given by

$$I = \frac{P_{\text{in}} e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}}.$$

The pressure difference in the capacitor is

$$P_{\text{out}} e^{i(\omega t + \phi)} = \frac{P_{\text{in}} e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \frac{1}{i\omega C} = \frac{P_{\text{in}} e^{i\omega t}}{i\omega C R - \omega^2 LC + 1}.$$

The amplitude is

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}.$$

To be smaller than  $P_{\text{in}}$ , for  $\omega \rightarrow 0$ :

$$(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2 R^2 > 0.$$

Replacing the expressions for  $L$ ,  $C$ , and  $R$  we get:  $\frac{64\eta^2 \ell^2}{3Ehr^3 \rho} > 1$ .

**A.3**
2.0pt

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}.$$

Condition:

$$\frac{64\eta^2 \ell^2}{3Ehr^3 \rho} > 1.$$

Alternative way to obtain  $P_{\text{out}}$ :

The amplitude of the current in the equivalent circuit is  $I_0 = \frac{P_{\text{in}}}{Z}$ , where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

$$P_{\text{out}} = \frac{1}{\omega C} \times I_0 = \frac{P_{\text{in}}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}.$$

## A.4

The previous condition can also be expressed as

$$h < \frac{64\eta^2 \ell^2}{3Er^3 \rho}.$$

For the network referred to in **A.2**

$$h < \frac{64\eta^2 \ell_0^2 \times 2^i}{3 \times 2^{2i/3} Er_0^3 \rho} = \frac{64 \times (3.5 \times 10^{-3})^2 \times (2.0 \times 10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0 \times 10^{-5})^3 \times 1.05 \times 10^3} \times 2^{i/3} = 7.7 \times 10^{-5} \times 2^{i/3}.$$

For  $i = 0$ , in the worse case scenario,

$$h_{\max} = 7.7 \times 10^{-5} \times 2^0 = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from  $18 \mu\text{m}$  to  $60 \mu\text{m}$ . A wall width smaller than  $80 \mu\text{m}$  is certainly reasonable.

**A.4** Maximum  $h = 8 \times 10^{-5} \text{ m}$

0.7pt

## Part B. Tumor growth (5.5 points)

### B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{cases} M_T = V_T \rho_T = V_T \rho_0 \left(1 + \frac{p}{K_T}\right) \\ M_N = V \rho_0 = (V - V_T) \rho_0 \left(1 + \frac{p}{K_N}\right) \end{cases}$$

The pressure,  $p$ , can be expressed as

$$p = \frac{M_T K_T}{V_T \rho_0} - K_T$$

and, then, used in the equation for  $M_N$ :

$$M_N = (V - V_T) \frac{M_N}{V} \left[ \left(1 - \frac{K_T}{K_N}\right) + \frac{M_T V K_T}{V_T M_N K_N} \right]$$

Simplifying and rearranging the terms, the equation for  $v$  becomes

$$(1 - \kappa) v^2 - (1 + \mu) v + \mu = 0,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to  $v = 0$  for  $\mu = 0$ )

**B.1**

$$v = \frac{1 + \mu - \sqrt{(1 + \mu)^2 - 4\mu(1 - \kappa)}}{2(1 - \kappa)}.$$

1.0pt

### B.2

For  $r < R_T$ , the conservation of energy implies that

$$4\pi r^2 (-k) \frac{dT}{dr} = \mathcal{P} \frac{4}{3} \pi r^3.$$

Therefore, the temperature difference to  $37^\circ\text{C} = 310.15\text{ K}$ ,  $\Delta T(r)$ , is given by

$$\Delta T(r) = -\frac{\mathcal{P}r^2}{6k} + C,$$

where  $C$  is a constant.

For  $r > R_T$ , the conservation of energy implies that

$$4\pi r^2(-k)\frac{dT}{dr} = \mathcal{P}\frac{4}{3}\pi R_T^3.$$

Therefore, the temperature difference to  $37^\circ\text{C}$  is

$$\Delta T(r) = \frac{\mathcal{P}R_T^3}{3kr}.$$

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at  $r = R_T$  gives

$$C = \frac{\mathcal{P}R_T^2}{2k}.$$

Therefore the temperature at the centre of the tumour, in SI units, is

<b>B.2</b> Temperature: $310.15 + \frac{\mathcal{P}R_T^2}{2k}$ .	1.7pt
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## B.3

The increase in temperature at the tumour surface (the lower temperature in the tumour) is

$$\Delta T(R_T) = \frac{\mathcal{P}R_T^2}{3k}.$$

This increase should be equal to 6.0 K. Therefore,

$$\mathcal{P} = \frac{3\Delta T k}{R_T^2} = \frac{3 \times 6 \times 0.6}{0.05^2} = 4.3 \text{ kW/m}^3.$$

<b>B.3</b> $\mathcal{P}_{\min} = 4.3 \text{ kW/m}^3$ .	0.5pt
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## B.4

We can relate  $\delta r$  with the pressure in the tumour, using the relation given in the text up to leading order in  $p - P_{\text{cap}}$ :  $\delta r = \frac{p - P_{\text{cap}}}{2(p_c - P_{\text{cap}})} \delta r_c$ . Therefore, if  $p - P_{\text{cap}}$  is very small, also it is  $\delta r$ .

The pressure can be related with the volume. We know that

$$\frac{M_N}{V_N} = \frac{\rho_0 V}{V - V_T} = \frac{\rho_0}{1 - v} = \rho_0 \left(1 + \frac{p}{K_N}\right).$$

And so  $p = \frac{K_N v}{1-v}$ .

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

$$\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left( \sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{i/3}} + \frac{2^{4(N-1)/3}}{2^{N-1} 2^{(N-1)/3} \left(1 - \frac{\delta r}{r_0/2^{(N-1)/3}}\right)^4} \right)$$

$$\Rightarrow \Delta P \simeq (Q_0 + \delta Q_0) \frac{\Delta P}{NQ_0} \left( N - 1 + 1 + \frac{4\delta r}{r_{N-1}} \right)$$

Noting that  $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$ , we obtain

$$1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{1}{1 + \frac{4\delta r}{Nr_{N-1}}} \simeq 1 - \frac{4\delta r}{Nr_{N-1}}.$$

And so:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}}.$$

Putting all together

**B.4**

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{2}{N} \frac{K_N v - (1-v)P_{\text{cap}}}{(1-v)(p_c - P_{\text{cap}})} \frac{\delta r_c}{r_{N-1}}.$$

2.3pt