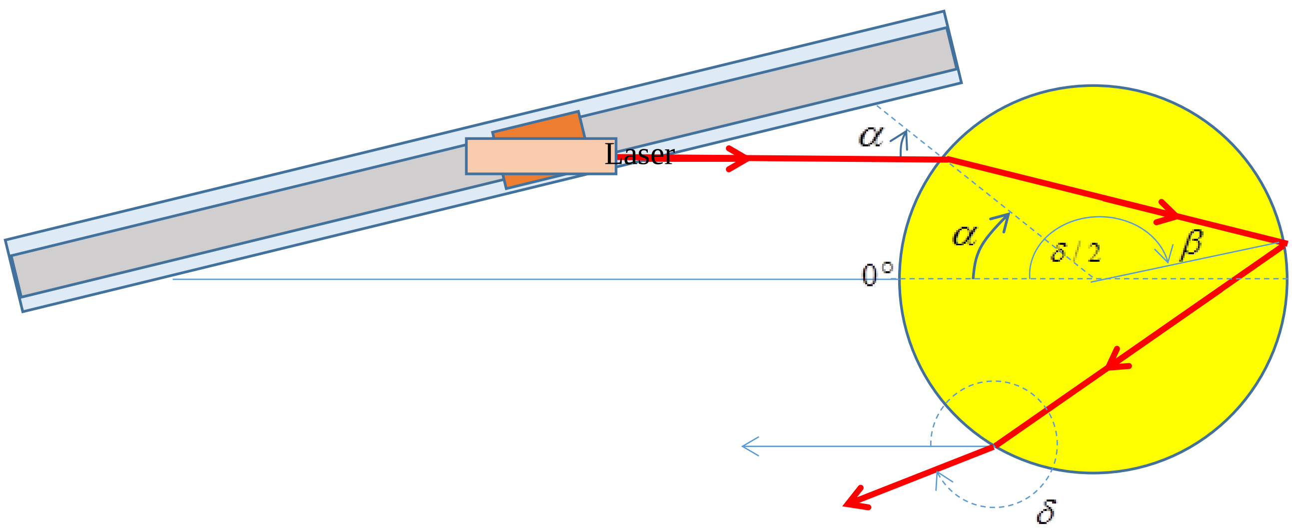
Optical Measurements – Solution

Part A: The refractive index of a disk

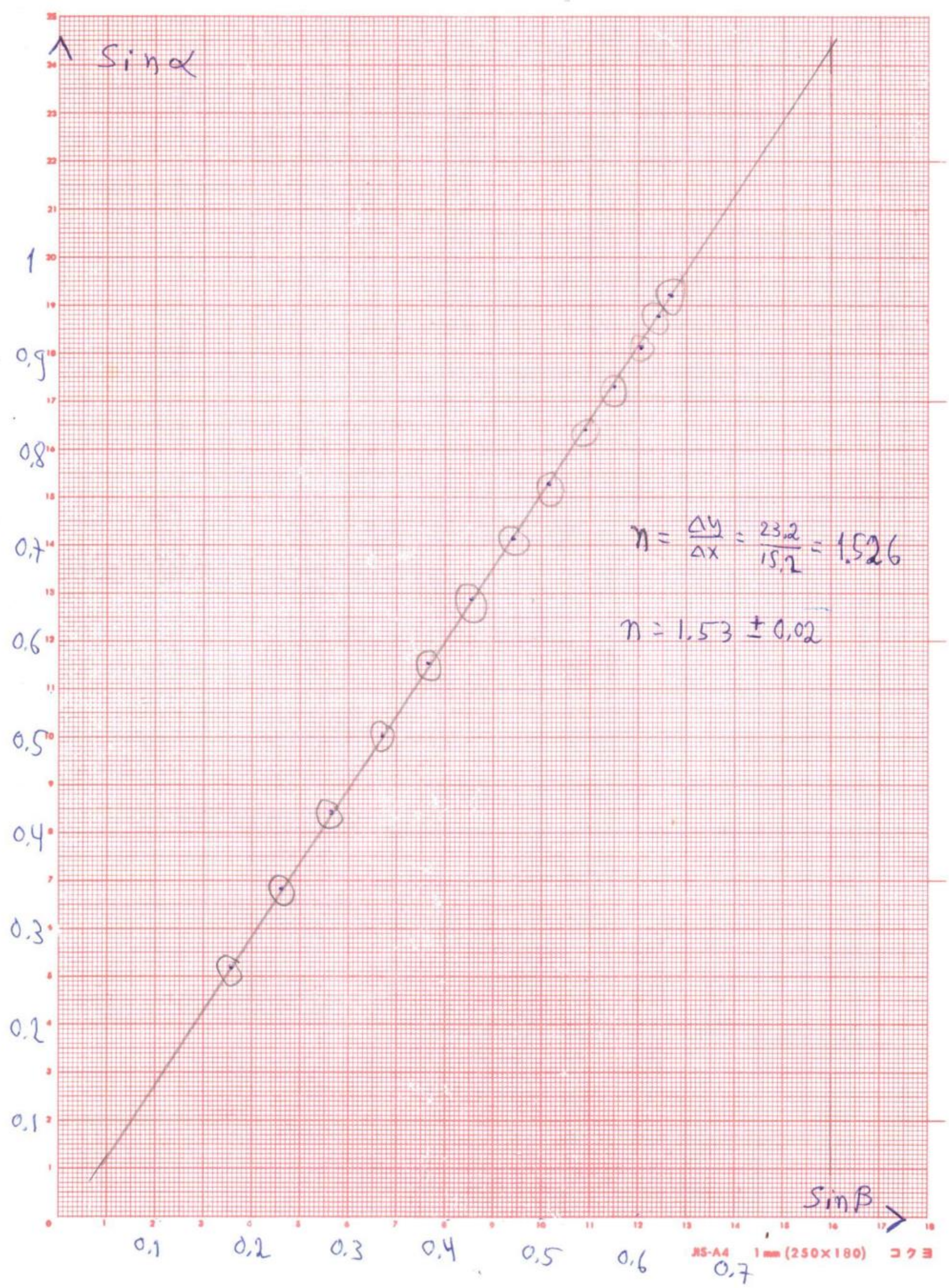
A.1: A sketch of the experimental setup for $N = 3$



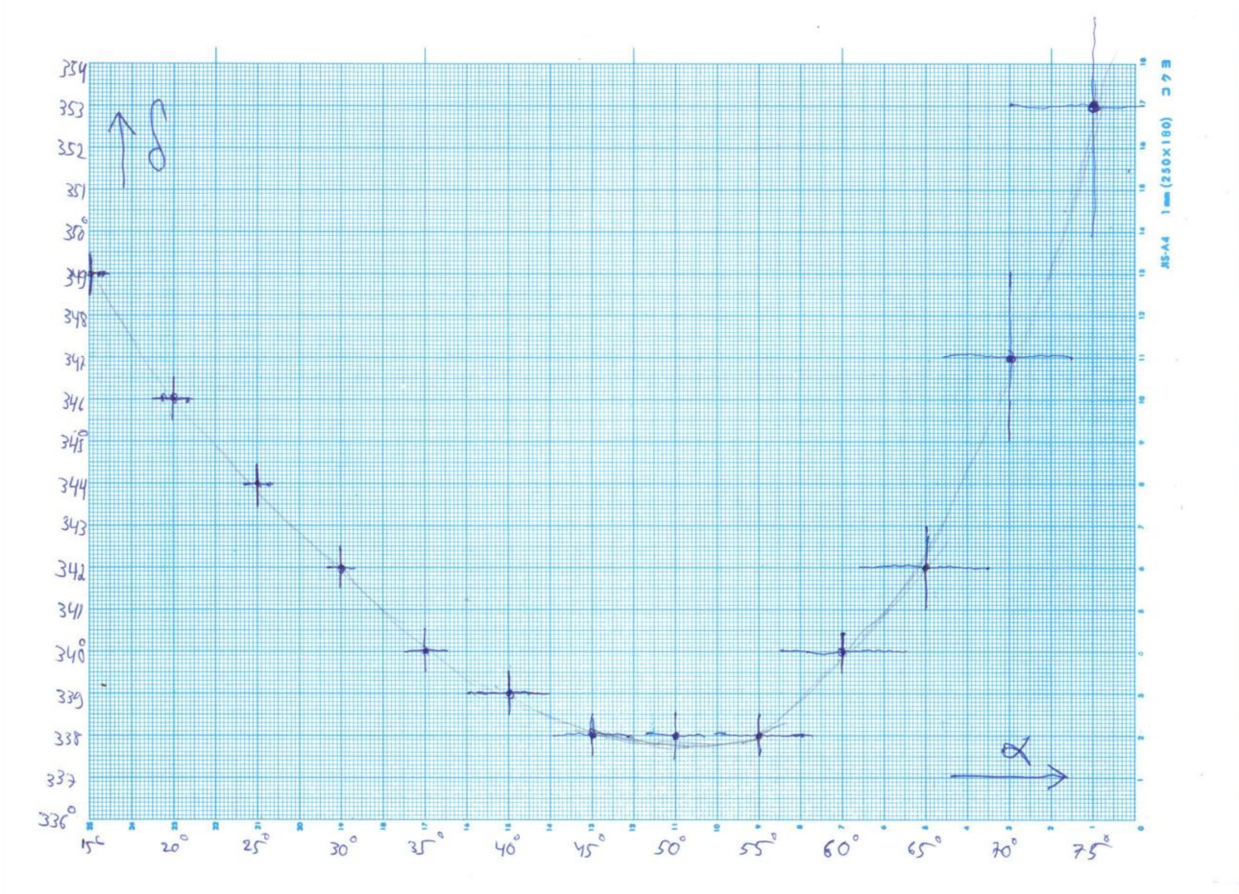
A.1: table of measured and calculated values

$\alpha(\circ)$	$\Delta\alpha(\circ)$	$\delta/2(\circ)$	$\Delta\delta/2(\circ)$	$\delta(\circ)$	$\Delta\delta(\circ)$	$\beta(\circ)$	$\sin\alpha$	$\sin\beta$
15	0.25	174.5	0.25	349	0.5	10.25	0.259	0.178
20	0.25	173	0.25	346	0.5	13.5	0.342	0.233
25	0.25	172	0.25	344	0.5	16.5	0.423	0.284
30	0.25	171	0.25	342	0.5	19.5	0.500	0.334
35	0.5	170	0.25	340	0.5	22.5	0.574	0.383
40	1	169.5	0.25	339	0.5	25.25	0.643	0.427
45	1	169	0.25	338	0.5	28	0.707	0.469
50	1	169	0.25	338	0.5	30.5	0.766	0.508
55	1	169	0.25	338	0.5	33	0.819	0.545
60	1.5	170	0.25	340	0.5	35	0.866	0.574
65	1.5	171	0.5	342	1	37	0.906	0.602
70	1.5	173.5	1	347	2	38.25	0.940	0.619
75	2	176.5	1.5	353	3	39.25	0.966	0.633

A.2:


Sin α
Sin β
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0,1
0,2
0,3
0,4
0,5
0,6
0,7
n = Δy/Δx = 23,2/15,2 = 1,526
n = 1,53 ± 0,02
JIS-A4 1mm (250×180) コクヨ

A.3:



By observing the remote screen, it is possible to identify the point in which δ is minimal at the highest accuracy.

The values we find are

$\alpha = 49° \pm 0.25°$ and $\delta = 338° \pm 0.5°$

A.4:

When δ is minimal, $\frac{d\delta}{d\alpha} = 0$.
Differentiating the relation $\delta = 2\alpha + (N-1)(180° - 2\beta)$ by α we get:

$2 - 2(N-1)\dfrac{d\beta}{d\alpha} = 0$ and therefore $\dfrac{d\beta}{d\alpha} = \dfrac{1}{N-1}$.

By differentiating Snell's law $\sin\alpha = n\sin\beta$ we get $\cos\alpha = n\cos\beta \cdot \frac{d\beta}{d\alpha} = \frac{n\cos\beta}{N-1}$

Squaring this result, as well as Snell's law and summing the expressions we get:

$$1 = \sin^2\alpha + \cos^2\alpha = n^2\sin^2\beta + \frac{n^2\cos^2\beta}{(N-1)^2}$$

Hence: $\frac{1}{n^2} = \sin^2\beta + \frac{\cos^2\beta}{(N-1)^2}$

We got an explicit relation between the refraction angle β and the refraction index of the material. Due to the multiple reflections inside the disk it is possible, by following all the point in which the beam hits the disk-air interface, to measure the angle β at very high accuracy.

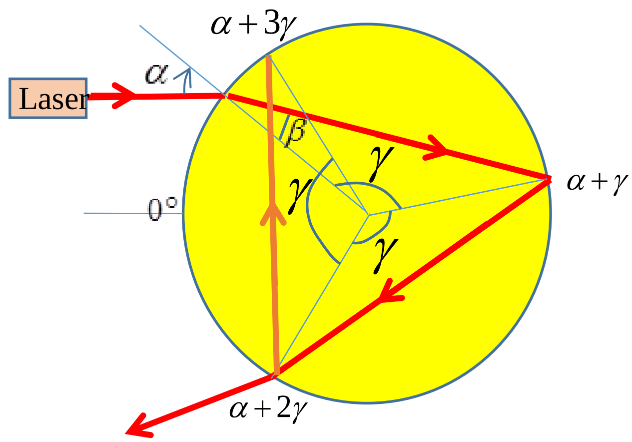
A.5: a sketch showing all the measured quantities:

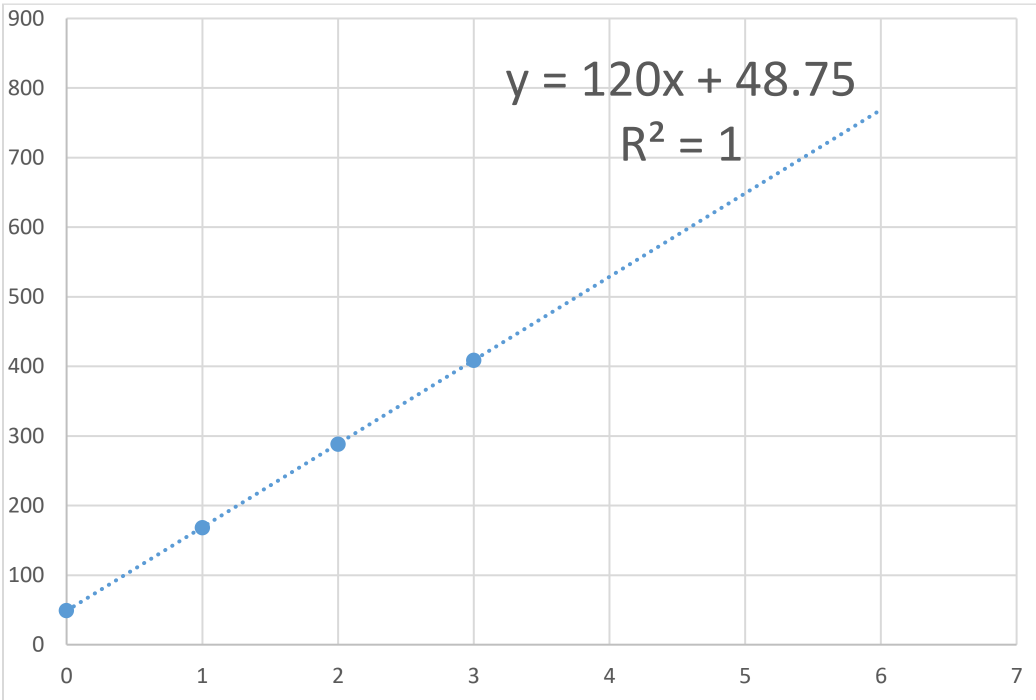
Define the angle $\gamma = 180° - 2\beta$, as shown in the sketch. In fact, after two reflections inside the disk the beam exits at a point very close to the entering point. We will measure the angular location of the points where the beam hits the interface after k reflection, for as many values of k as we can:

k	$\alpha + k\gamma$
0	49
1	168.5
2	288.5
3	409

Note: for the case of $N = 3$ it is not possible to measure for $k > 3$ as in this case, starting from $k = 3$ the impact points co-inside with previous points.

Next we draw a graph of $y = \alpha + k\gamma$ vs. k and find the linear regression slope, γ:



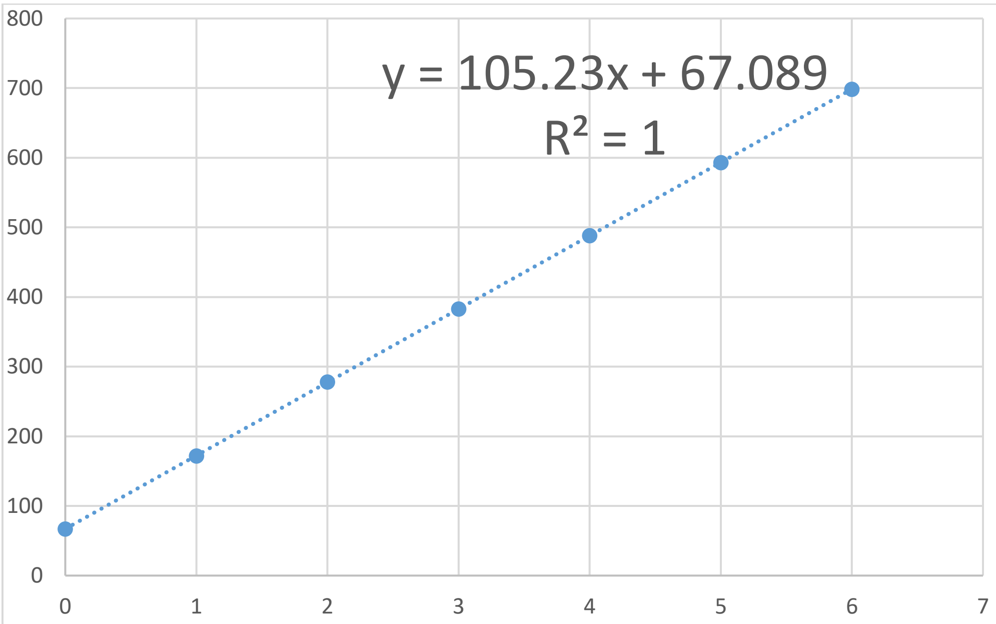


From $\gamma = 120°$ we get $\beta = 30°$, and using the equation we derived in A.4 we get:

$$n = \frac{1}{\sqrt{(\sin\beta)^2 + (\cos\beta)^2/(N-1)^2}} = 1.512$$

A.6: We will identify the beam exiting the disk after 4 refractions/reflections $(N = 4)$ and we will change the incident angle until we get δ_{min} for $N = 4$. We will measure $\alpha + k\gamma$ as a function of the number of times the beams hits the disk-air interface, k:

k	$\alpha + k\gamma$
0	67
1	172
2	278
3	383
4	488
5	593
6	698.5

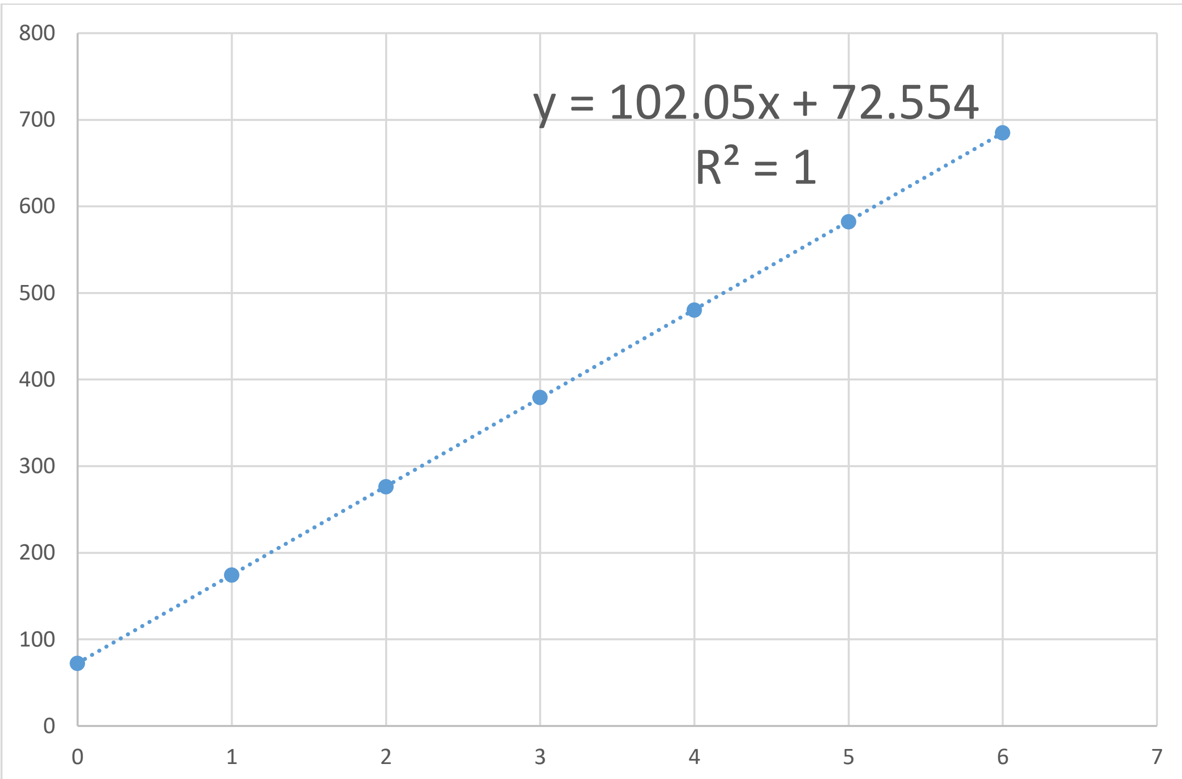


$$n = \frac{1}{\sqrt{(\sin\beta)^2 + (\cos\beta)^2/(N-1)^2}} = 1.511$$

We'll repeat this process for $N = 5$:

We will identify the beam exiting the disk after hitting the disk-air interface 5 times ($N = 5$) and measure $\alpha + k\gamma$ as a function of the number of hits, k:

k	$\alpha + k\gamma$
0	72.5
1	174.5
2	276.5
3	379.5
4	480.5
5	582.5
6	685

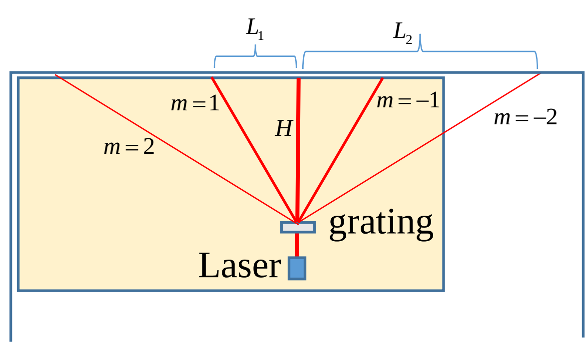


$$n = \frac{1}{\sqrt{(\sin\beta)^2 + (\cos\beta)^2/(N-1)^2}} = 1.519$$

Averaging the three results we get: $n = \dfrac{1.519+1.511+1.512}{3} = 1.514 \pm 0.004$

Section B – parameters of a diffraction grating

B.1: We will mark on the table a point Q, at a distance of about $H = 70cm$ from the screen – the wall of the experimental chamber - and at an equal distance from the chamber's side walls. Using the given measuring tape we will mark on the screen two points P_1 and P_2, at an equal distance of about 100cm from the left and form the right of the marked point Q. On the screen, we will mark a point P, placed in the middle of the interval P_1P_2. Then, we will aim a laser to go through the points QP. This beam will be perpendicular to the wall that will be used as a screen.



Standard method:
We will place the grating such that the beam passes through it. By gently rotating the grating we will make sure that diffraction ordered 1 and -1 as well as 2 and -2 will appear in symmetrically around the zero order point. Note that the position of the zero order on the screen does not depend on the angle α. In this situation it ok to assume that the incident angle of the beam on the grating is $\alpha = 0$.
As in the sketch, we will measure H, L_1 and L_2 and use the relation $d\sin\theta_m = m\lambda$.
The measured values are $2L_1 = 53.3\mathrm{cm}$, $2L_2 = 163.5\mathrm{cm}$ and $H = 60.8\mathrm{cm}$.

For the first order we get $\frac{\lambda}{d} = 0.4015$. For the second order we get $\frac{\lambda}{d} = 0.4012$.

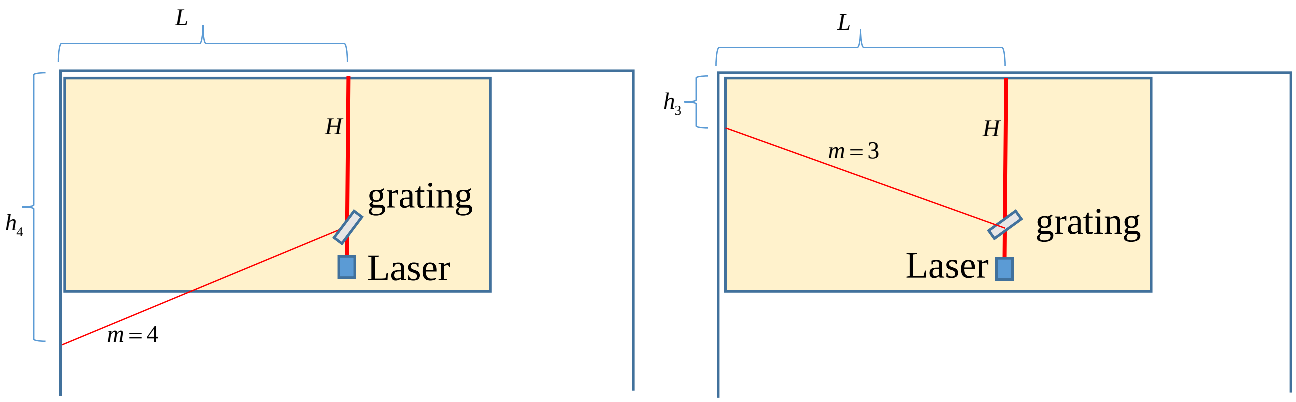
B.2: A second method
Getting higher orders is not possible at an incident angle of $\alpha = 0$. Thus we will change α and as a result the angle θ_m will change. There is an angle in which θ_m is minimal. By differentiating the relation $d\left(\sin\alpha + \sin(\theta_m - \alpha)\right) = m\lambda$ by α we get that at the minimum ($\frac{d\theta_m}{d\alpha} = 0$) one gets $\cos\alpha - \cos(\theta_m - \alpha) = 0 \Rightarrow \alpha = \frac{\theta_m}{2}$. From this we get $2d\sin(\frac{\theta_m}{2}) = m\lambda$.

Note that there is no need to measure the angle α, but rather to identify, by changing α, the minimum of θ_m.

Using this method it is possible to measure also ordered $m = 1$ and $m = 2$. For $m = 2$ and $m = -2$ we can verify that the beam is perpendicular to the screen by making sure the distance of these two ordered from the zero order is identical.

For $m = 3$, we will change α to get θ_{3min} and measure the distances L and h_3.

The measured values, as shown in the sketch below, are $H = 67.0\,\text{cm}$, $L = 100.2\,\text{cm}$, $h_3 = 37.8\,\text{cm}$.



We get $\tan\theta_{3\min} = \dfrac{L}{H - h3} = \dfrac{100.2}{67.0 - 37.8} = 3.432$ and hence $\theta_{3\min} = 73.75°$

Therefore: $\dfrac{\lambda}{d} = \dfrac{2}{3}\sin\dfrac{\theta_{3\min}}{2} = \dfrac{2}{3}\sin\dfrac{73.75°}{2} = 0.400$

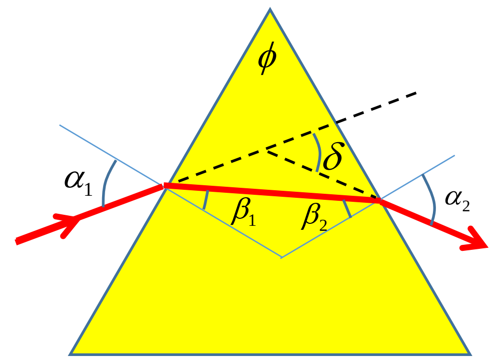
For $m = 4$ we will change α to get $\theta_{4\min}$ and measure the distance h_4.

The measured values are $H = 67.0\,\text{cm}$, $L = 100.2\,\text{cm}$, $h_4 = 96.3\,\text{cm}$.

From the sketch we get $\tan(\theta_{4\min} - 90°) = \dfrac{h_4 - H}{L} = \dfrac{96.3 - 67.0}{100.2} = 0.2924$

Hence $\theta_{4\min} = 106.3°$, therefore $\dfrac{\lambda}{d} = \dfrac{2}{4}\sin\dfrac{\theta_{4\min}}{2} = \dfrac{1}{2}\sin\dfrac{106.3°}{2} = 0.400$

Section C – the refraction index of a triangular prism



C.1: From the sketch showing the path of the laser beam and from the principle the beam path reversal we get that the deflection angle δ from the direction of in the incoming beam will not change if we switch the angles α_1 and α_2. Thus we get that δ achieves an extremum value (in fact, a minimal value) when the situation is perfectly symmetric, that is when $\alpha_1 = \alpha_2$. In this case,

$$\beta_1 = \beta_2 = \frac{\phi}{2}.$$

For the symmetric case, the incident angle α holds the relation $\alpha = \frac{\delta}{2} + \frac{\phi}{2}$ and from Snell's law we get $\sin\left(\frac{\delta}{2} + \frac{\phi}{2}\right) = n \sin\frac{\phi}{2}$.

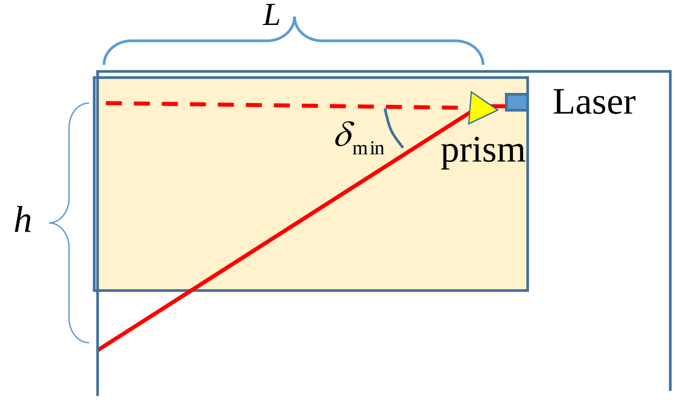
If the prism is not exactly equilateral, we will mark the angles of the prism by $\phi_i = 60° + 2\varepsilon_i$. From the sum of angles in a triangle we get $\sum \varepsilon_i = 0$. Additionally $\beta_i = 30 + \varepsilon_i$. In this case $\delta_{\min} = \delta_0 + 2\Delta_i$ where δ_0 is the minimal δ when $\phi = 60°$.

From Snell's law we get $\sin\left(\frac{\delta_0}{2} + 30° + \Delta_i + \varepsilon_i\right) = n \sin(30° + \varepsilon_i)$. Making the small angle approximation: $\sin\left(\frac{\delta_0}{2} + 30°\right) + \cos\left(\frac{\delta_0}{2} + 30°\right)(\Delta_i + \varepsilon_i) = n \sin 30° + n \cos 30° \cdot \varepsilon_i$

From the equation that holds for $60°$ prism we get $\cos\left(\frac{\delta_0}{2} + 30°\right)(\Delta_i + \varepsilon_i) = n \cos 30° \cdot \varepsilon_i$

Averaging for all three angles we get $\langle \Delta_i \rangle = 0$, and therefore $n = 2\sin\left(\frac{\langle \delta_{\min} \rangle}{2} + 30°\right)$

C.2. We will use the full length of the table to magnify the distances as much as possible. We will build the setup, as described in the sketch, so that in the absence of the prism, the laser beam will hit the screen (the chamber's wall) perpendicularly. We will attach the prism holder base to the table using the adhesive tape. On it we will place the prism holder and the prism itself. We will rotate the prism to find the minimal deflection angle δ_{min}. We will then repeat the measurement of δ_{min} for each corner of the prism.



The measured values are given in the table:

Corner No.	L	h	$\delta_{\min}$
1	141.6 ± 0.2 cm	175.2 ± 0.3 cm	$51.05° \pm 0.1°$
2	141.0 ± 0.2 cm	167.1 ± 0.3cm	$49.84° \pm 0.1°$
3	140.7 ± 0.2 cm	171.4 ± 0.3cm	$50.62° \pm 0.1°$

Calculation of the error in $\delta_{\min}$:

$$\tan\delta_{\min} = \frac{h}{L} \Rightarrow \frac{1}{\cos^2\delta_{\min}}\Delta\delta_{\min} = \sqrt{\left(\frac{\Delta h}{L}\right)^2 + \left(\frac{h\Delta L}{L^2}\right)^2}$$

Therefore, $\Delta\delta_{\min} = \cos^2\delta_{\min}\sqrt{\left(\frac{\Delta h}{L}\right)^2 + \left(\frac{h\Delta L}{L^2}\right)^2}$

Substituting the measured values we get

$$\Delta\delta_{\min} = \cos^2 51.05°\sqrt{\left(\frac{0.3}{141.6}\right)^2 + \left(\frac{175.2\cdot 0.2}{141.6^2}\right)^2} = 0.0017rad = 0.1°$$

The error in the average value of the two angles is

$$\Delta\langle\delta_{\min}\rangle = \frac{0.1°}{\sqrt{3}} = 0.06° = 1\cdot 10^{-3}rad$$

From the table we get that the average value of $\delta_{\min}$ is $\langle \delta_{\min} \rangle = 50.50°$

Therefore the refraction index of the prism is

$$n = 2\sin\left(\frac{\langle \delta_{\min} \rangle}{2} + 30°\right) = 2\sin\left(\frac{50.50°}{2} + 30°\right) = 2\sin 55.25° = 1.6433$$

And the error in n : $\Delta n = 2\cos 55.25° \cdot 0.5\Delta\langle \delta_{\min} \rangle = \cos 55.25° \cdot 1 \cdot 10^{-3} = 6 \cdot 10^{-4}$

Thus: $n = 1.6433 \pm 0.0006$

As the laser wavelength may vary between lasers up to a standard deviation of $\pm 10\,\text{nm}$, the value found in the literature is $n(\lambda \pm \Delta\lambda) = 1.6425 \pm 0.0007$.

Optical Measurements – Marking Scheme

Part A: The refractive index of a disk

A.1	drawing diagram:	
	The ruler, beam, disk, and α appear in the diagram (1 missing item: 0.1 pts; more than 1 missing items: 0 pts)	0.2 pts
	the incoming beam parallel to the diameter through $0°$ (can be identified by the position of α)	0.1 pts
	ruler is tilted with respect to the beam	0.1 pts
	at least 10 measurement points (for 8-9 points: 0.2 pts; for 6-7 points: 0.1 pts; less: 0 pts) (without Δs: 10 points: 0.2 pts; 8-9 points: 0.1 pts; less: 0 pts)	0.3 pts
	full 15-75 degrees region (25-65 degrees region: 0.1 pts)	0.2 pts
	varying $\Delta\delta$ according to the spot size on the screen	0.1 pts
A.2	calculated β from δ and α for all rows in the table	0.1 pts
	calculated $\sin\alpha$ and $\sin\beta$ for all rows in the table	0.1 pts
	at least 8 measured points appear in the graph	0.1 pts
	the data covers at least 75% of each coordinate length	0.1 pts
	there are labels in each axis	0.1 pts
	plotted regression line and calculate slope	0.1 pts
	value of n: $1.50 \le n \le 1.53$ if $1.48 \le n < 1.50$ or $1.53 < n \le 1.55$: 0.1 pts	0.3 pts
	value of $0.005 \le \Delta n \le 0.03$, if $1.45 \le n \le 1.58$; otherwise: 0 pts	0.1 pts
A.3	The graph includes a minimum angle of δ	0.1 pts
	labels in each axis and error bars of $\Delta\delta$ appear	0.1 pts
	value of $\delta_{\min}$: $336° \le \delta_{\min} \le 338°$	0.2 pts

	$335° \le \delta_{\min} < 336°$ or $338° < \delta_{\min} \le 339°$: 0.1 pts	
	value of $\alpha(\delta_{\min})$ $49° \le \alpha(\delta_{\min}) \le 51°$	0.1 pts
A.4	Stated that $\frac{d\delta}{d\alpha} = 0$	0.1 pts
	found that $\frac{d\beta}{d\alpha} = \frac{1}{N-1}$ (N = 3: full points)	0.1 pts
	Snell's Law $\cos\alpha = \frac{n\cos\beta}{N-1}$ or eq. (N = 3: full points)	0.2 pts
	get $\frac{1}{n^2} = \sin^2\beta + \frac{\cos^2\beta}{(N-1)^2}$ or eq. with α (N = 3: full points)	0.3 pts
A.5	figure includes ray path and measured angles	0.1 pts
	measurement of $\phi_j = \alpha + j\gamma$ for $j = 0,1,2,3$ only for j = 0,3: 0.2 pts only for j = 0,1,2 : 0.2 pts only for j = 0,2: 0.1 pts	0.3 pts
	plotted a graph of ϕ_j vs. j	0.1 pts
	found value for β (or γ)	0.1 pts
	value of n: $1.510 \le n \le 1.520$ $1.505 \le n < 1.510$ or $1.520 < n \le 1.525$: 0.1 pts	0.2 pts
A.6	for $N = 4$:	
	measurement of $\phi_j = \alpha + j\gamma$ for $j = 0,1,2,...,6$ (7 values) only for $j = 0$ and $j = 5$ or 6: 0.2 pts only for $j = 0$ and $j = 3$: 0.1 pts	0.3 pts
	plotted graph of ϕ_j vs. j	0.1 pts
	found value for β (or γ)	0.1 pts
	value of n: $1.510 \le n \le 1.520$	0.2 pts

	$1.505 \le n < 1.510$ or $1.520 < n \le 1.525$: 0.1 pts	
	For $N = 5$:	
	measurement of $\phi_j = \alpha + j\gamma$ for $j = 0,1,2,\ldots,6$ (7 values)	0.3 pts
	measurement of $\phi_j = \alpha + j\gamma$ for $j = 0$ and $j = 5$ or 6	0.2 pts
	measurement of $\phi_j = \alpha + j\gamma$ for $j = 0$ and $j = 4$	0.1 pts
	plotted graph of ϕ_j vs. j	0.1 pts
	found value for β (or γ)	0.1 pts
	value of n: $1.510 \le n \le 1.520$ $1.505 \le n < 1.510$ or $1.520 < n \le 1.525$: 0.1 pts	0.2 pts
	value of $\langle n \rangle$: $1.512 \le \langle n \rangle \le 1.518$	0.1 pts

Part B: The parameters of a diffraction grating

In part B, the final results of each student should be rescaled relative to the reference of $\lambda/d = 0.400$, according to the table supplied separately, using the ID of the grating recorded by the student in his/her answer sheet.

B.1	Plotted a diagram with all of the requested items	0.1 pts
	Distance of the diffraction grating from the screen > 45 cm	0.1 pts
	for $m = 1$ value of λ/d: $0.395 \le \lambda/d \le 0.405$ if $0.39 \le \lambda/d < 0.395$ or $0.405 \le \lambda/d < 0.41$: 0.1 pts	0.2 pts
	for $m = 2$ value of λ/d: $0.395 \le \lambda/d \le 0.405$ if λ/d: $0.39 \le \lambda/d < 0.395$ or $0.405 \le \lambda/d < 0.41$: 0.1 pts	0.3 pts
B.2	First method – using extremum:	
	plotted the diagram with all requested items	0.1 pts
	the ray is definitely not perpendicular to the grating	0.1 pts
	the grating angle changes between $m = 3$ and $m = 4$ or $m = 4$ maximum is backwards	0.1 pts
	showed that the minimal angle is at $\alpha = \theta/2$	0.5 pts

	for extremum calculation with error: 0.4 pts	
	value of $\theta_{3\min}$: $73.0° \le \theta_{3\min} \le 74.5°$ or $36.5° \le \alpha_{3\min} \le 37.25°$ if $72.0° \le \theta_{3\min} \le 75.5°$ or $36.0° \le \alpha_{3\min} \le 37.75°$: 0.1 pts (around θ_{min} satisfying $2\sin\frac{\theta_{min}}{2} = m\frac{\lambda}{d}$ for the grating ID; $\theta_{3min} = 73.74° + 214.86° \times \Delta; \Delta = \lambda/d - 0.4$)	0.3 pts
	for $m=3$ value of λ/d : $0.395 \le \lambda/d \le 0.405$ if λ/d : $0.39 \le \lambda/d < 0.395$ or $0.405 \le \lambda/d < 0.41$: 0.1 pts	0.2 pts
	value of $\theta_{4\min}$: $105.5° \le \theta_{4\min} \le 107.0°$ or $52.25° \le \alpha_{4\min} \le 53.5°$ if $104.0° \le \theta_{4\min} \le 108.5°$ or $52.0° \le \alpha_{4\min} \le 54.25°$: 0.1 pts (around θ_{4min} satisfying $2\sin\frac{\theta_{min}}{2} = m\frac{\lambda}{d}$ for the grating ID; $\theta_{4min} = 106.26° + 381.97° \times \Delta; \Delta = \lambda/d - 0.4$)	0.3 pts
	for $m=4$ value of λ/d : $0.395 \le \lambda/d \le 0.405$ if λ/d : $0.39 \le \lambda/d < 0.395$ or $0.405 \le \lambda/d < 0.41$: 0.1 pts	0.2 pts
	Alternative method – measuring α directly:	
	plotted the diagram with all requested items	0.1 pts
	measuring α	0.3 pts
	for $m=3$ value of λ/d : $0.395 \le \lambda/d \le 0.405$ if λ/d : $0.39 \le \lambda/d < 0.395$ or $0.405 \le \lambda/d < 0.41$: 0.3 pts	0.7 pts
	for $m=4$ value of λ/d : $0.395 \le \lambda/d \le 0.405$ if λ/d : $0.39 \le \lambda/d < 0.395$ or $0.405 \le \lambda/d < 0.41$: 0.3 pts	0.7 pts

Part C: The refractive index of a triangular prism

C.1	understood that $\delta_{\min} = \delta_{\text{sym}}$, or independently obtained $49.5° \le \delta_{\text{sym}} \le 51.5°$ in C.2	0.4 pts
C.2	measured $\delta_{\min}$ for at least one prism angle: $49.5° \le \delta_{\min} \le 51.5°$	0.3 pts
	measured $\delta_{\min}$ for two more prism angles: $49.5° \le \delta_{\min} \le 51.5°$	0.3 pts
	distance between prism and screen larger than 120 cm	0.1 pts

	finding $\langle\delta_{\min}\rangle$: $50.30° \le \langle\delta_{\min}\rangle \le 50.70°$	0.3 pts
	making correct calculation of $\Delta\langle\delta_{\min}\rangle$, $\Delta\langle\delta_{\min}\rangle \le 0.1°$	0.1 pts
	finding n: $1.641 \le n \le 1.644$ $1.640 \le n < 1.641$ or $1.644 < n \le 1.645$: 0.3 pts $1.639 \le n < 1.640$ or $1.645 < n \le 1.646$: 0.2 pts $1.637 \le n < 1.639$ or $1.646 < n \le 1.648$: 0.1 pts	0.4 pts
	finding Δn using correct $\Delta\delta_{\min}$, $\Delta n \le 0.001$	0.1 pts