Zero-length springs and slinky coils – Solution

Part A: Statics

A.1 The force F causes the spring to change its length from L_0 to L. Since equal parts of the spring are extended to equal lengths, we get: $\frac{\Delta y}{\Delta l} = \frac{L}{L_0} \to \Delta y = \frac{L}{L_0}\Delta l$.

Since $L = \max\left\{\frac{F}{k}, L_0\right\}$, we get $\Delta y = \max\{\frac{F}{kL_0}\Delta l, \Delta l\}$. From this result we see that any piece of length Δl the spring behaves as a ZLS with spring constant $k^* = k\frac{L_0}{\Delta l}$.

A.2 Let us compute the work of the force. From Task A.1: $dW = F(x)dx = \frac{kL_0}{\Delta l}xdx$.
Hence, $\Delta W = \int_{\Delta l}^{\Delta y} \frac{kL_0}{\Delta l} xdx = \frac{kL_0}{\Delta l}\frac{x^2}{2}\Big|_{\Delta l}^{\Delta y} = \frac{kL_0}{2\Delta l}(\Delta y^2 - \Delta l^2)$.

A.3. At every point along the statically hanging spring the weight of the mass below is balanced by the tension from above. This implies that at the bottom of the spring there is a section of length l_0 whose turns are still touching each other, as their weight is insufficient to exceed the threshold force kL_0 to pull them apart. The length l_0 can be derived from the equation:
$\frac{l_0}{L_0}Mg = kL_0$, hence $l_0 = \frac{kL_0^2}{Mg} = \alpha L_0$.

For $l > l_0$, a segment of the unstretched spring between l and $l + \mathrm{d}l$ feels a weight of $\frac{l}{L_0}Mg$ from beneath, which causes its length to stretch from $\mathrm{d}l$ to $dy = \frac{F}{kL_0}dl = \frac{l}{L_0}Mg\frac{dl}{kL_0} = \frac{Mg}{kL_0^2}ldl = \frac{l}{l_0}dl$.

Integration of the last expression over the stretched region, up to the point L_0, gives its height when the spring is stretched

$$H = l_0 + \int_{l_0}^{L_0} \frac{l}{l_0}dl = l_0 + \frac{l^2}{2l_0}\Bigg|_{l_0}^{L_0} = l_0 + \frac{1}{2l_0}(L_0^2 - l_0^2) = \frac{L_0^2}{2l_0} + \frac{l_0}{2} = \frac{L_0}{2}\left(\alpha + \frac{1}{\alpha}\right)$$

Part B: Dynamics

B.1. From Task A.3 we have $H(l) = \frac{l^2}{2l_0} + \frac{l_0}{2}$. We now calculate the position of the center of mass of the suspended spring. The contribution of the unstretched section of height l_0 at the bottom, having a mass of $\frac{l_0}{L_0}M = \alpha M$, is $\alpha M \frac{l_0}{2}$. The position of the center of mass is obtained by summing the contributions of its elements:

$$H_{cm} = \frac{1}{M}\left[\frac{l_0}{2}\alpha M + \int_{l_0}^{L_0} H(l)dm\right] = \frac{1}{M}\left[\frac{\alpha L_0}{2}\alpha M + \int_{l_0}^{L_0}\left(\frac{l^2}{2l_0} + \frac{l_0}{2}\right)\frac{Mdl}{L_0}\right]$$
$$= \frac{\alpha^2 L_0}{2} + \frac{1}{L_0}\left[\frac{l^3}{6l_0} + \frac{l_0}{2}l\right]_{l_0}^{L_0} = \frac{\alpha^2 L_0}{2} + \frac{1}{L_0}\left[\frac{L_0^3 - l_0^3}{6l_0} + \frac{l_0}{2}(L_0 - l_0)\right]$$

Where we have used $dm = \frac{dl}{L_0}M$. Substituting $l_0 = \alpha L_0$ yields

$$H_{cm} = L_0\left[\frac{1}{6\alpha} - \frac{\alpha^2}{6} + \frac{\alpha}{2}\right]$$

When the spring is contracted to its free length L_0, its center of mass is located at $\frac{L_0}{2}$. From the falling of the center of mass at acceleration g we get:

$$\frac{g}{2}t_c^2 = H_{cm} - \frac{L_0}{2} = L_0\left[\frac{1}{6\alpha} - \frac{\alpha^2}{6} + \frac{\alpha}{2} - \frac{1}{2}\right] = \frac{L_0}{6\alpha}(1-\alpha)^3$$

Hence, $t_c = \sqrt{\frac{L_0}{3g\alpha}(1-\alpha)^3}$.

For $k = 1.02$ N/m, $L_0 = 0.055$ m, $M = 0.201$ kg, and $g = 9.80$ m/s^2, we have $\alpha = 0.0285$, and $t_c = 0.245$ s.

B.2. The moving top section of the spring is pulled down by its own weight, $m_{top}g = Mg\frac{(L_0 - l)}{L_0}$ and also by the tension in the spring below, which is equal to the weight Mgl/L_0 of the stationary section of the spring. Thus, the moving top section experiences a constant force $F = Mg$ throughout its whole fall. Another way to see that, is that a total force of Mg is exerted on the spring, but only the moving part experiences it. Let's calculate the position of the center of mass at equilibrium of the upper part, i.e., all points with $l' > l$ for some $l > l_0$. From part A, the position of a small portion $\Delta l'$ with coordinate l' is: $H(l') = \frac{l'^2}{2l_0} + \frac{l_0}{2}$ and the center of mass of this part is:

$$H_{cm-upper-i} = \frac{L_0}{M(L_0-l)}\int_l^{L_0} H(l')dm = \frac{L_0}{M(L_0-l)}\int_l^{L_0}\left(\frac{l'^2}{2l_0}+\frac{l_0}{2}\right)dm$$

$$= \frac{L_0}{M(L_0-l)}\int_l^{L_0}\left(\frac{l'^2}{2l_0}+\frac{l_0}{2}\right)\frac{Mdl'}{L_0} = \frac{1}{(L_0-l)}\int_l^{L_0}\left(\frac{l'^2}{2l_0}+\frac{l_0}{2}\right)dl'$$

$$= \frac{1}{(L_0-l)}\left[\frac{l'^3}{6l_0}+\frac{l_0 l'}{2}\right]_l^{L\_0} = \frac{L_0^2+L_0 l+l^2}{6l_0}+\frac{l_0}{2}$$

The position of the upper part of CM when it contracts to a length $L_0 - l$ is $H_{cm-upper-f} = \frac{l^2}{2l_0} + \frac{l_0}{2} + \frac{1}{2}(L_0 - l)$. The change in the CM during the contraction process is: $\Delta H_{cm-upper} = H_{cm-upper-i} - H_{cm-upper-f} = \frac{L_0^2+L_0 l-2l^2}{6l_0} - \frac{1}{2}(L_0-l) = \frac{(L_0-l)(L_0+2l)}{6l_0} - \frac{1}{2}(L_0-l)$.

The acceleration of the CM of the upper part is $a_{CM} = \frac{FL_0}{M(L_0-l)} = \frac{gL_0}{L_0-l}$.
From the work energy theorem we get the equation $v_{upper-f}^2 = 2a_{CM}\Delta H_{cm-upper}$, hence

$$v_{upper-f}^2 = 2\frac{gL_0}{L_0-l}\left[\frac{(L_0-l)(L_0+2l)}{6\alpha L_0} - \frac{1}{2}(L_0-l)\right] = 2g\left[\frac{L_0+2l}{6\alpha} - \frac{1}{2}L_0\right]$$

$$= \frac{2g}{3\alpha}l + \left(\frac{1}{3\alpha}-1\right)gL_0$$

Therefore, $A = \frac{2g}{3\alpha}$ and $B = \left(\frac{1}{3\alpha}-1\right)gL_0$.

Note that for $l = L_0$, we have $v_{upper-f}^2 = L_0 g\frac{1-\alpha}{\alpha}$ and for $l = l_0 = \alpha L_0$, we get $v_{upper-f}^2 = L_0 g\frac{1-\alpha}{3\alpha}$, hence, the moment we release the spring its velocity is finite (not zero, the meaning is that it accumulate this velocity in time that is much shorter than the contracting time t_c) and it decreases to $\frac{1}{\sqrt{3}}$ of the initial value when $l = l_0$.

B.3. Note that even though the center of mass of the spring accelerates downwards constantly, the moving top section actually decelerates, while the position of the center of mass moves down the spring. The speed of the top section $v(l)$, calculated in Task B2, decreases and approaches the value $\sqrt{A\alpha L_0 + B}$ immediately before it attaches to the bottom section of height $l_0 = \alpha L_0$, which was unstretched and at rest. Once the moving top section attaches to the resting bottom section, its momentum is shared between both sections, so the speed further decreases just before the whole spring starts accelerating downwards as a single mass. Thus, the minimum speed is that of the whole spring immediately after its full collapse. From momentum conservation, we have

$$Mv_{min} = m_{top}v(l_0) = M\left(1 - \frac{l_0}{L_0}\right)\sqrt{A\alpha L_0 + B}$$

$$v_{min} = (1-\alpha)\sqrt{A\alpha L_0 + B}$$

Part C: Energetics

C.1. From the moment the spring is released, the acceleration of its center of mass is governed by the external force Mg and therefore the gravitational potential energy of the spring is fully converted into the kinetic energy of the center of mass of the spring, which just before hitting the ground is equal to the kinetic energy of the spring.

All that is left is the elastic energy stored in the spring, which is converted into heat, sound, etc. To calculate it, we consider the elastic energy stored in a segment dh of the stretched spring, which when unstretched lies between l and $l + \mathrm{d}l$, using the result of Task A.2, $\Delta W = \frac{kL_0}{2\Delta l}(\Delta l_2^2 - \Delta l^2)$, by choosing $\Delta l = dl$ and $\Delta l_2 = dy$, and using $dy = \frac{l}{l_0}dl$ (which was obtained in Task A.3), we get:

$dW = \frac{kL_0}{2}\left(\frac{l^2}{l_0^2} - 1\right)dl$. Integrating from l_0 to L_0 we find

$$W = \int_{l_0}^{L_0} \frac{kL_0}{2}\left(\frac{l^2}{l_0^2} - 1\right)dl = \frac{kL_0}{2}\left[\frac{l^3}{3l_0^2} - l\right]_{l_0}^{L_0} = \frac{kL_0}{2}\left(\frac{L_0^3 - l_0^3}{3l_0^2} - (L_0 - l_0)\right)$$

$$= \frac{kL_0^2}{2}\left(\frac{1-\alpha^3}{3\alpha^2} - (1-\alpha)\right) = \frac{kL_0^2}{6\alpha^2}(1-\alpha)^2(2\alpha+1)$$

$$= MgL_0\frac{(1-\alpha)^2(2\alpha+1)}{6\alpha}$$

Zero-length springs and slinky coils – Marking Scheme

Part A: Statics

A.1	First method:	
	realized that $\frac{\Delta y}{\Delta l}=\frac{L}{L_0}$	0.2 pts
	realized that $L=\frac{F}{k}$	0.1 pts
	mentioned that $\Delta y=\Delta l$ if $F<kL_0$	0.1 pts
	correct result	0.1 pts
	Second method:	
	realized $k^*=k\frac{L_0}{\Delta l}$	0.2 pts
	mentioned that $\Delta y=\Delta l$ if $F<kL_0$	0.1 pts
	used $F-F_0=k^*(\Delta y-\Delta l)$	0.1 pts
	correct result	0.1 pts
A.2	used $F(x)=\frac{kL_0}{\Delta l}x$	0.1 pts
	used $\Delta W=\int_{\Delta l}^{\Delta l_2}\frac{kL_0}{\Delta l}xdx$ (if $\Delta W=\frac{kL_0}{\Delta l}(\Delta y-\Delta l)^2$ – no pts.)	0.3 pts
	correct result	0.1 pts
A.3	calculating l_0, the length of the closed turns	0.2 pts
	correct result for l_0	0.3 pts
	realized $F(l)=Mg\frac{l}{L_0}$	0.3 pts

	expressed $dy = \frac{Mg}{kL_0^2} l dl$	0.2 pts
	used $H = l_0 + \int_{l_0}^{L_0} \frac{Mg}{kL_0^2} l dl$	0.3 pts
	(if assumed $l_0 = 0$)	0.1 pts
	correct result expressed using the required variables	0.7 pts
	correct result but not expressed using the required variables	0.4 pts

Part B: Dynamics

B.1	used $H(l) = \frac{l^2}{2l_0} + \frac{l_0}{2}$	0.3 pts
	the contribution of element dl to the center of mass: $\left(\frac{l^2}{2l_0} + \frac{l_0}{2}\right)\frac{Mdl}{L_0}$	0.2 pts
	considering the unstretched part	0.2 pts
	$H_{\text{cm}} = \frac{1}{M}\left[\frac{l_0}{2}\alpha M + \int_{l_0}^{L_0} H(l) dm\right]$ (first term 0.1, second term 0.3)	0.4 pts
	correct calculation of $H_{\text{cm}} = L_0\left[\frac{1}{6\alpha} - \frac{\alpha^2}{6} + \frac{\alpha}{2}\right]$	0.4 pts
	found the displacement of the center of mass: $\Delta H = H_{\text{cm}} - \frac{L_0}{2}$	0.2 pts
	used $\frac{g}{2} t_c^2 = H_{\text{cm}} - \frac{L_0}{2}$	0.2 pts

	correct result expressed using the required variables	0.5 pts
	correct result but not expressed using the required variables	0.4 pts
	numerical value of time	0.1 pts
B.2	Default method – work of external force equals the change in kinetic energy	
	calculated the position of the upper part ($l < l' < L_0$) before the spring is released, used $H(l') = \frac{l'^2}{2l_0} + \frac{l_0}{2}$	0.2 pts
	$H_{cm-upper} = \frac{L_0}{M(L_0 - l)} \int_l^{L_0} H(l')dm = \frac{L_0}{M(L_0 - l)} \int_l^{L_0} \left(\frac{l'^2}{2l_0} + \frac{l_0}{2} \right) \frac{Mdl'}{L_0}$	0.3 pts
	understood that the external force is Mg	0.3 pts
	found $H_{cm-upper} = \frac{L_0^2 + L_0 l + l^2}{6l_0} + \frac{l_0}{2}$	0.5 pts
	found the position of the center of mass of the moving part: $H_{cm-upper-f} = \frac{l^2}{2l_0} + \frac{l_0}{2} + \frac{1}{2}(L_0 - l)$	0.3 pts
	$\Delta H_{cm-upper} = H_{cm-upper} - H_{cm-upper-f} = \frac{(L_0 - l)(L_0 + 2l)}{6l_0} - \frac{1}{2}(L_0 - l)$	0.3 pts
	using $Mg\Delta H_{cm-upper} = \frac{M(L_0 - l)}{L_0} v^2$	0.3 pts
	obtained $v_{upper-f}^2 = \frac{2g}{3\alpha} l + \frac{L_0 g}{3\alpha} - L_0 g$	0.3 pts
	Other methods: There are other ways to solve this part	1.3 pts

	Finding A and B without deriving $v_{\mathrm{I}}(l)=\sqrt{Al+B}$	
B.3	realized that the velocity decreases in time when a stationary part still exists	0.1 pts
	substituted $l=l_0$ to find $v_{\mathrm{I}}(l_0)=\sqrt{Al_0+B}$	0.1 pts
	used conservation of momentum (plastic collision): $Mv_{\min}=m_{top}v(l_0)$	0.2 pts
	correct result	0.1 pts

Part C: Energetics

C.1	realized that the loss of mechanical energy equals the elastic energy of the spring	0.4 pts
	use $dW=\frac{kL_0}{2dl}\left(\left(\frac{1}{l_0}ldl\right)^2-dl^2\right)=\frac{kL_0}{2}\left(\frac{l^2}{l_0^2}-1\right)dl$	0.4 pts
	calculated the elastic energy of the spring: $W=\int_{l_0}^{L_0}\frac{kL_0}{2}\left(\frac{l^2}{l_0^2}-1\right)dl$	0.4 pts
	correct result $W=MgL_0\frac{(1-\alpha)^2(2\alpha+1)}{6\alpha}$	0.3 pts