A calculation of the surface charges and the electric field outside steady current carrying conductors

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Abstract. The surface charges and the electric field outside steady current carrying conductors has been for long the subject of many investigations. It turns out that both quantities strongly depend on the configuration of the circuit. In this work an attempt to distinguish and clarify the essential factors of this dependence is presented. To this end, a simple, three-dimensional model is proposed and quantitative results are derived which are in agreement with experimental data. In addition, the existence of electric fields outside current carrying conductors and their behaviour can be easily demonstrated in class by recourse to the model, with the help of a computer program, thus avoiding inconvenient or dangerous experimental set-ups.

Zusammenfassung. Die Flächenladung und das elektrische Feld außerhalb eines gleichstromtragenden Leiters ist seit langem das Thema vieler Untersuchungen gewesen. Es zeigt sich, daßbeide Größen stark von der Struktur des Stomkreises abhängen. In diesem Werk wird der Versuch, die wesentlichen Faktoren dieser Abhängigkeit zu unterscheiden und zu erklären, presentiert. Zu diesem Zweck wird ein einfaches dreidimensionales Modell vorgeschlagen und quantitative Ergebnisse, die in Übereinstimmung mit experimentellen Daten stehen, werden abgeleitet. Zusätzlich kann die Existenz elektrischer Felder außerhalb stromtragender Leiter und ihr Verhalten durch dieses Modell und mit Hilfe eines Computerprogramms einfach im Klassenzimmer demonstriert werden, so daßumständliche und gefährliche Experimentaufbauten vermieden werden können.

1. Introduction

One of the most common assumptions in physics is to consider the electric field inside a Current Carrying Conductor (CCC) homogeneous. This intuitively acceptable assumption inevitably entails the existence of surface charge densities on the CCCs when the laws of electrostatics are taken into account [9]. This surface charge is usually omitted in textbooks [2, 17, 23], apart from rare exceptions [8, 24]. This fact could lead to the misconception that a 'closed current in a stationary conductor exerts no force on stationary electricity' [13] or in other words that electric fields outside CCCs do not exist. However, experiments that were made [4, 7, 14] showed that this is not the case. A charge density does really exists on the surface of a CCC and the outside electric field is different from zero though very small [20]. Nevertheless, the results from previous theoretical works depend strongly on the circuit configuration, giving a vague picture of the whole effect.

The first theoretical publication [9] in the subject is relatively old (1941) and it predicts the formation of surface charges on the CCC, but in a very peculiar geometry; the current of the CCC returns to the battery by a conductor which is coaxial to the CCC. Later on (1969), Sarachman [22] tried to find the electric field outside a CCC by general principles, like energy conservation. It turned out [10] however, that the form of the charge density depends strongly on the imposed boundary conditions. This form is determined by the solution of a related boundary value problem. Different boundary conditions result in different forms of the surface charge density. In the case of parallel CCCs it was pointed out that the surface charge density should vary linearly with the distance [21]. Additionally, two different aspects of charge density formation in CCCs had also been suggested [3, 11]. Up to that time (1970) the battery region of the circuit had attracted no interest. A fruitful debate started after Professor Eugen Merzbacher set again the problem [5, 6, 12, 25] a decade later. As an outcome, Heald [5] was the first to find the electric field outside closed circuits, where the battery

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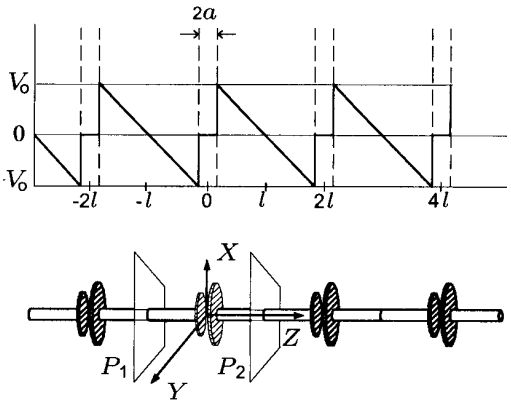

Figure 1. The infinite periodic array of batteries and cylindrical resistors, along with the voltage variation across the circuit elements. (Notice that no energy flows through the planes P_1 and P_2, which define a 'unit cell' of the periodic array.)

region was included. Heald noticed the importance of the circuit geometry and for simplification he analysed only two-dimensional models. Recently, another model was proposed [1] where a completely different surface charge density was used. Apparently, a clarification and classification of the essential parameters (battery region, geometry, boundary conditions, etc) of the problem is needed.

The clarification of the essential parameters, which is among the objectives of the present work, is attempted through a new model circuit. It is necessary, for the general case that such a model satisfy at least the following requirements:

- The CCC should be three dimensional.
- The potential at infinity should be taken zero.
- Both the battery and the CCC should be considered, since the role of the battery region and the related voltage drop along the resistive wire have to be included.
- The model should be free of any particular circuit geometry, like bending of the wires or changes of the wires' cross section, since such complications restrict the application of the results.

On this basis, one can derive a fundamental contribution to the formed surface charge density. Other contributions, due to the circuit geometry, can be considered as corrections to it. In principle, all contributions should be taken into account [1, 3] in order to estimate the form and the magnitude of the total surface charge density.

2. The model

Imagine an infinite periodic array of batteries and cylindrical resistors that are directed along the z-axis (see figure 1). Such a model, apart that it meets the requirements of the previous paragraph, satisfies another two as well:

(i) Each battery balances the voltage drop along the wire before it so that no source at infinity is needed [9, 25].
(ii) There is no energy flow between the planes normal to the conductor in the middle of two successive resistors†.

So we don't have to take into account the rest of the array but doing physics on this energetically isolated part of the space (the period) is enough. We will focus our interest in the area shown on figure 1 for the sake of simplicity. Both the battery and the conductor are considered as cylindrical cylinders of radius R. Inside the cylinder the potential is a periodic odd function of the z coordinate with half period l(see figure 1). According to our model this potential U_0 is described within the first half period by:

$$U_0(z) = V_0 \begin{cases} 0 & \text{if } z \le \alpha l \\ \dfrac{l-z}{(1-\alpha)l} & \text{if } z > \alpha l \end{cases} \tag{1}$$

where α is the portion of the half period which corresponds to the battery. Such a potential simulates both the voltage variation in the battery region of a Weston cell [17] and the voltage drop along the resistor.

Since the boundary condition (1) doesn't involve the angle ϕ neither does the potential over all space. Consequently the Laplace equation for our problem in cylindrical coordinates can be written as:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2}$$

If we substitute a separable function $\Phi(\rho, z) = P(\rho)Z(z)$ in (2) we get:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2 Z = 0 \tag{3}$$

$$\frac{\mathrm{d}^2 P}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho} - k^2 P = 0 \tag{4}$$

where k is the separation constant. Owing to the fact that our boundary condition (1) is periodic and odd only the sine solutions of (3) are acceptable. Since $Z(z)$ is a sine function, the separation constant is real and equation (4) is the modified Bessel equation of order zero. From the two possible solutions I_0 and K_0 of the modified Bessel equation only K_0 satisfies the usual boundary condition that the potential is zero at infinity. So the solution of (2) in our case takes the form:

$$\Phi(\rho, z) = \sum_k A_k K_0(k\rho)\sin(kz) \tag{5}$$

† Planes P_1, P_2 of figure 1.

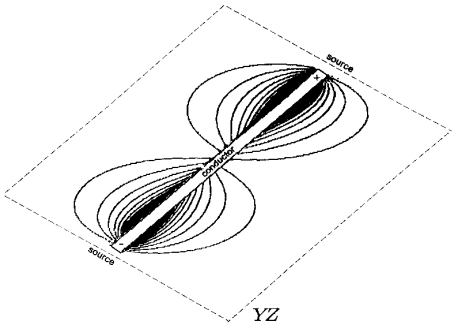

Figure 2. The intersection of the equipotential surfaces and the *YZ* plane. The region shown is that of the resistive wire with the two battery terminals at the edges. The equipotential surfaces join points on the batteries with points on the conductor.

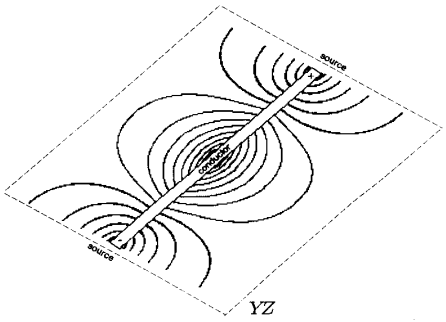

Figure 3. The electric field lines on the *YZ* plane. The region shown is the same as in figure 2.

2.1. The potential

Choosing the constants A_k so that (5) satisfies (1) at $\rho = R$, we get the potential:

$$V(\rho, z) = \sum_{n=1}^{\infty} \frac{K_0(\frac{n\pi\rho}{l})}{K_0(\frac{n\pi R}{l})} a_n \sin(\frac{n\pi z}{l}) \tag{6}$$

with:

$$a_n = V_0 \left(\frac{\cos(n\pi\alpha)}{n\pi} - \frac{2\sin(n\pi\alpha)}{n^2\pi^2(\alpha - 1)} \right) \tag{7}$$

for the space outside the wire, $\rho > R$. Inside the CCC the electric field is constant, as it was stated from the beginning. In the case of the Weston cell this holds for the battery too. Thus, for $\rho < R$ the potential is $v(\rho, z) = U_0(z)$. A graph of the equipotentials from (6) is shown on figure 2.

2.2. The electric field outside the CCC

Since there is energy flow outside the CCC the electric field there, is non-zero. The electric field $\vec{E} = -\nabla V$ outside the CCC is expressed in cylindrical coordinates, with the help of (6), as follows:

$$E_\rho = \vec{E} \cdot \vec{e}_\rho = \sum_{n=1}^{\infty} \frac{K_1(\frac{n\pi\rho}{l})}{K_0(\frac{n\pi R}{l})} \frac{n\pi a_n}{l} \sin(\frac{n\pi z}{l}) \tag{8}$$

$$E_z = \vec{E} \cdot \vec{e}_z = -\sum_{n=1}^{\infty} \frac{K_0(\frac{n\pi\rho}{l})}{K_0(\frac{n\pi R}{l})} \frac{n\pi a_n}{l} \cos(\frac{n\pi z}{l}) \tag{9}$$

The electric field lines are shown in figure 3. It is clearly seen that the electric field is not parallel to the resistive wire. Therefore a surface charge density should exist on the conductor. In the area of space where $z \in [-\alpha l, \alpha l]$ and $|r| > R$, that includes points outside the battery and away from the conductor, the electric field is parallel to the axis of the cylinder in agreement with the results of Heald [5].

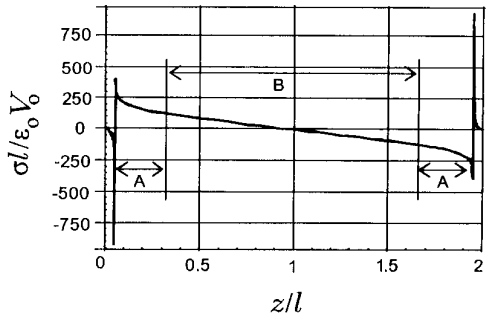

Figure 4. The dimensionless surface charge density $(\frac{\sigma l}{\epsilon_0 V_0})$ as a function of z/l. The region shown, [0,2], encloses the resistive wire in the middle, so as to stress the characteristic linear dependence of σ (region B) as opposed to its singular behaviour near the battery terminals (region A),(α = 0.05, $R/l = 10^{-3}$).

2.3. The surface charge density

The surface charge density on the surface of the conductor is given by:

$$\sigma(z) = \epsilon_0 \vec{E} \cdot \vec{n} \tag{10}$$

where ϵ_0 is the vacuum dielectric constant and $\vec{n}$ the normal to the cylinder surface, directed outwardly from the cylinder. So the surface charge density is according to (8):

$$\sigma(z) = \epsilon_0 \sum_{n=1}^{\infty} \frac{K_1(\frac{n\pi R}{l})}{K_0(\frac{n\pi R}{l})} \frac{n\pi a_n}{l} \sin(\frac{n\pi z}{l}) \tag{11}$$

Plotting (11) on figure 4 we can distinguish between two characteristic regions, denoted after A and B in the figure.

In the region A of the conductor, near the plates of the battery, a large amount of charge is gathered. Indeed, near the plates of the battery and in the absence of the conductor there are fringing phenomena. When the conductor is connected to the battery a homogeneous electric field is finally established inside it. So extra surface charge should gather in this region

to homogenise the fringing electric field produced by the plates of the battery. The surface charge density shows a singular behaviour near the plates of the battery, which is a result found previously for the two-dimensional circuits [5]. No matter the singularity, the total charge formed can be calculated easily because the related integral converges.

In the region B of the conductor, away from the battery, a linearly z-dependent surface charge density is formed. This result has already been considered [21] and is related once again to the homogeneity of the electric field inside the CCC. The charges at the region A produce an electric field that far from them and inside the wire has a non-zero radial component. Far away from these charges and near the axis of the CCC this radial component is small and to the lowest order proportional to z, exactly like the field produced by a charged disk [17]. We know, however, [14] that the charges on the surface of the CCC arrange themselves to homogenize the electric field inside the conductor. Hence the surface charge density in region B has to be linearly dependent so as to produce an electric field equal in measure to the one it has to balance.

Thus the surface charge density in the case of our model is arranged in two characteristic formations: a singular one near the plates of the battery (region A) and a linear one away from the battery (region B). These formations of surface charge density exist in a realistic circuit, no matter the geometry, and we shall try to find quantitative results about them.

2.4. The energy flow.

Another interesting physical quantity the form of which depends crucially on the surface charge density is the energy flow. The surface charge density (11) and the charges at the plates of the battery are responsible for the potential (6) which determines the energy flow in the following manner. The equipotential surfaces are the surfaces on which the field lines of the Poynting vector lie [5, 8]. Actually , the Poynting vector $\vec{S} = (\vec{E} \times \vec{B})/\mu_0$ is normal to the electric field vector which as the gradient of the potential is normal to the equipotential surfaces. So the field lines of the Poynting vectors are embedded in the equipotential surfaces. Therefore, from the graph of the equipotentials (figure 2) we can clearly see that the energy flows from the battery region to the resistive wire through the outer space [15, 18]. The fact that no energy flows between two successive planes (figure 1) simplifies the picture and clarifies the fact that each battery in the array is responsible for the voltage drop in the wire included in the energy isolated region.

2.5. The amount of charge on the surface of a CCC.

As we can see from the graph (figure 4) of the charge density (11) an amount of charge Q sets on the surface of the cylinder near the positive terminal of the source and an amount of charge $-Q$ sets on the surface of the cylinder near the negative terminal. Q is calculated from relation (11):

$$Q = \int_{\alpha l}^{l} \int_{0}^{2\pi} \sigma(z) R \mathrm{d}\phi \mathrm{d}z \tag{12}$$

Doing this integration we get:

$$Q = 2\pi\epsilon_0 R \sum_{n=1}^{\infty} \frac{K_1(\frac{n\pi R}{l})}{K_0(\frac{n\pi R}{l})} a_n(\cos(\alpha\pi n) - (-1)^n) \tag{13}$$

An important quantity referred to the charge formed on the surface of the conductor is $\kappa = Q_0/(V_0 l)$. This is the surface charge per unit length and voltage drop along the wire:

$$\kappa = 2\pi\epsilon_0 \frac{R}{l} \sum_{n=1}^{\infty} \frac{K_1(\frac{n\pi R}{l})}{K_0(\frac{n\pi R}{l})} \frac{a_n}{V_0}(\cos(\alpha\pi n) - (-1)^n) \tag{14}$$

If we plot κ as a function of the ratio R/l and α (fig.5), it turns out that it has a small variation when the ratio R/l covers a range of six orders of magnitude, from 10^{-8} to 10^{-2} and α covers the realistic range, from 10^{-4} to 10^{-1} . From the values of κ, which amount some tenths of million electrons per metre and voltage drop, we can see that the whole effect is very small. In fact, the experiments that showed the existence of such charges on CCCs where made [7, 14] by using 10 kV, low-current, power supplies so as to have 10^{11} electrons per metre, or better a few nC per metre. Such charge densities, though very small, can line up grass seeds [7] or minute plastic fibers [14] and make their existence detectable.

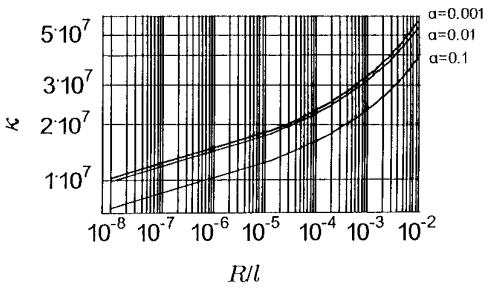

Figure 5. The κ-parameter as a function of R/l. The relative length of the battery, α, parametrizes the curves. (Notice that, although both R/l and α vary over many orders of magnitude, κ is relatively insensitive.)

3. Analysis of other surface charge formations on CCCs

The surface charge density so far discussed is longitudinal and consists of the two characteristic formations mentioned above. It gives rise [19] to a transverse electric field that prevents current flow perpendicular to the wire. We will call it as *longitudinal* charge density. It is pertinent to this work to present

other mechanisms, previously appeared in the literature, that also produce surface charge densities.

Firstly, Chester [3] proposed that a *configurational* EMF should exist when the cross section of CCC changes abruptly. This configurational EMF corresponds to a dipole surface charge density at the point of discontinuity and its magnitude as estimated [3] is:

$$\delta V_c = \frac{m}{2e} v_{drift}^2 \tag{15}$$

where m is the mass of the charge carriers. Because of the smallness of the drift velocity in CCCs the dipole charge density that stems from this configurational EMF mechanism is extremely small. This configurational dipole charge density didn't exist in our model because there weren't any abrupt changes of the wire cross section.

Secondly, Matzek and Russell [11] proposed that since the homogeneous current density inside the conductor produces a magnetic field, a *Hall-type* potential difference should exist between the axis and the surface of a CCC. This potential difference is formed by a uniform [16] charge density inside the conductor whose magnitude is given by [16]:

$$\varrho = en \frac{v_{drift}^2}{c^2} \tag{16}$$

Rosser [20] calculated the Hall-type charge density in copper and found (for $I = 1$ A, $\pi R^2 = 10^{-3}\,\mathrm{m}^2$) that it is just 6000 electrons per cubic metre! So the Hall-type charge density is extremely small.

Thirdly, Rosser [20] proposed that an excess charge density exists due to the bending of CCC. He calculated that along the bending of a copper wire (with $R = 10^{-2}$ m) to form a right angle, only one hundred electrons were needed to guide a current of one hundred A. Of course, this charge is proportional to the current and for common magnitudes of currents is much less than one electron!

Finally, Aguirregabiria *et al* [1] considered a certain model circuit in which the current was generated by magnetic induction. In their case the electric field generated by the varying magnetic field had a perpendicular to the wire component and a *transverse* surface charge density had to exist to balance it. The charge density was transverse in the sense that, in addition to having a modulation along the current flow, it had also an angular modulation along ϕ. This source of charge density is clearly due to the geometry of the circuit and its order of magnitude is again small [1]:

$$\sigma(\phi) = 2\epsilon_0 E_\perp \sin\phi + O(R/l) \tag{17}$$

where $E_\perp$ is the external field perpendicular to the wire and $O(R/l)$ a function of R/l and higher orders. Clearly, such a surface charge density is absent in our model which is axially symmetric.

Summarizing, we can say that charge distributions in CCCs are due to five different mechanisms:

- The usual longitudinal surface charge density [9] which we calculated.
- The configurational dipole charge density [3].
- The Hall-type charge density [11].
- The excess charge density due to wire bending [20].
- The transversal surface charge density [1].

Notice that, in our model the configurational and the transversal charge densities do not exist because of the simple geometry, whereas the Hall-type charge density is taken into account implicitly by the solution of the Laplace equation for the space outside the wire.

4. Calculation of the surface charges in a realistic circuit

It becomes apparent from the previous analysis that the charge distribution in a CCC is generally determined by: (1) The dimensionality of the circuit, (2) The boundary conditions, (3) The voltage drop along the CCC, (4) The battery region, (5) The cross section of the CCC, (6) The curvature of the CCC, (7) The presence of external electric fields. The first four parameters were included in our model, exactly the same way they intervene in a realistic circuit. In the general case however, the surface charges on the conductors should be estimated by adding the other three parameters necessitated by the mechanisms explained in the previous paragraph. The main component is the longitudinal surface charge density whose magnitude can be described concisely by the value of the κ parameter. Hence, using the voltage drop along the conductor and its length we can find from figure 5 the amount of charge on its surface. Complicated geometry can be taken into account by the concepts of configurational EMF, excess charge density due to wire bending and transversal charge density. When the cross section of the conductor is varied, a dipole surface charge density, to which a configurational EMF can be related [3], is arranged near the variation. A very small amount of charge, however, can account even for abrupt such changes in good conductors. Moreover, currents going through a bending of the wire can be guided by few electrons [20] accumulated near the bending. Other geometrical asymmetries as well as external electric fields can be taken into account by the transverse surface charge density [1]. The surface charges of other circuit elements produce an 'external' field on the CCC under consideration which is balanced by the transverse surface charge density. These fields are in general small, smaller than or equal to 10^{-4} V m^{-1}, and cause a transversal variation of the surface charge density with an amplitude smaller than or equal to 10^4 electrons per square metre! Finally, the Hall-type charge density [11] is much smaller than the other charge densities and can be neglected for practical purposes.

5. Conclusions

The purpose of this work was to clarify the charge density formations in CCCs and calculate their

magnitude. We achieved to separate the major contribution of these charge densities via a certain model including an array of batteries and resistive cylindrical conductors. This contribution consists of a singular surface charge density near the plates of the source, or otherwise at points where fringing exists in the absence of the conductor, and of a linearly dependent surface charge density away from the source and along the conductor. The charge per unit metre and voltage drop that is formed on the surface of the conductor is denoted by the κ parameter. It turns out that for all practical applications κ is few tens of million electrons per metre and voltage drop. The κ parameter can be used even in the case of a complicated circuit geometry provided that other contributions concerning surface charge density [1, 3, 20] are taken into account. Typical orders of magnitude of these contributions are given.

Within the framework of the present analysis the electric field lines and the energy flow vector were also considered. The results compare favourably with the idea that energy flows in the conductors from the space around them [15, 18] and complement Heald's results [5] in the case of two-dimensional circuits. On the teaching aspect of the problem, the electric and Poynting vector field lines that show the existence of the charge density and the flow of energy from the source to the resistive wire can be easily reproduced by a computer program† that was prepared in accordance with the model. Thus experimental devices [7, 14], including high-voltage generators could be avoided when we demonstrate the phenomenon for teaching purposes.

† A PC-version of the program `electro.zip` is available by `ftp::anonymous` to the University of Athens supercomputer `atlas.uoa.ariadne-t.gr`.

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